Law of sines

In trigonometry, the **law of sines**, **sine law**, **sine formula**, or **sine rule** is an equation relating the lengths of the sides of any triangle to the sines of its angles. According to the law,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where a, b, and c are the lengths of the sides of a triangle, and α, β, and γ are the opposite angles (see figure 2), while R is the radius of the triangle's circumcircle. When the last part of the equation is not used, the law is sometimes stated using the reciprocals;

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

The law of sines can be used to compute the remaining sides of a triangle when two angles and a side are known—a technique known as triangulation. It can also be used when two sides and one of the non-enclosed angles are known. In some such cases, the triangle is not uniquely determined by this data (called the *ambiguous case*) and the technique gives two possible values for the enclosed angle.

Law of Sines

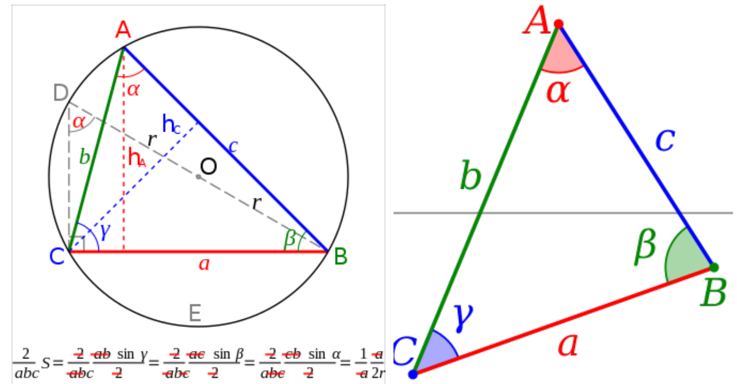


Figure 1, With circumcircle

Figure 2, Without circumcircle

Two triangles labelled with the components of the law of sines. α, β and γ are the angles associated with the vertices at capital A, B, and C, respectively. Lower-case a, b, and c are the lengths of the sides opposite them. (a is opposite α, etc.)

The law of sines is one of two trigonometric equations commonly applied to find lengths and angles in scalene triangles, with the other being the law of cosines.

The law of sines can be generalized to higher dimensions on surfaces with constant curvature.[1]

History

H.J.J. Wilson's book Eastern Science[2] states that the 7th century Indian mathematician Brahmagupta describes what we now know as the law of sines in his astronomical treatise Brāhmasphuṭasiddhānta. In his partial translation of this work, Colebrooke[3] translates Brahmagupta's statement of the sine rule as: The product of the two sides of a triangle, divided by twice the perpendicular, is the central line; and the double of this is the diameter of the central line.

According to Ubiratàn D'Ambrosio and Helaine Selin, the spherical law of sines was discovered in the 10th century. It is variously attributed to Abu-Mahmud Khojandi, Abu al-Wafa' Buzjani, Nasir al-Din al-Tusi and Abu Nasr Mansur.[4]

Ibn Muʿādh al-Jayyānī's *The book of unknown arcs of a sphere* in the 11th century contains the spherical law of sines.[5] The plane law of sines was later stated in the 13th century by Nasīr al-Dīn al-Tūsī. In his *On the Sector Figure*, he stated the law of sines for plane and spherical triangles, and provided proofs for this law.[6]

According to Glen Van Brummelen, "The Law of Sines is really Regiomontanus's foundation for his solutions of right-angled triangles in Book IV, and these solutions are in turn the bases for his solutions of general triangles."[7] Regiomontanus was a 15th-century German mathematician.

Proof

The area of any triangle can be written as one half of its base times its height. Selecting one side of the triangle as the base, the height of the triangle relative to that base is computed as the length of another side times the sine of the angle between the chosen side and the base. Thus depending on the selection of the base, the area T of the triangle can be written as any of:

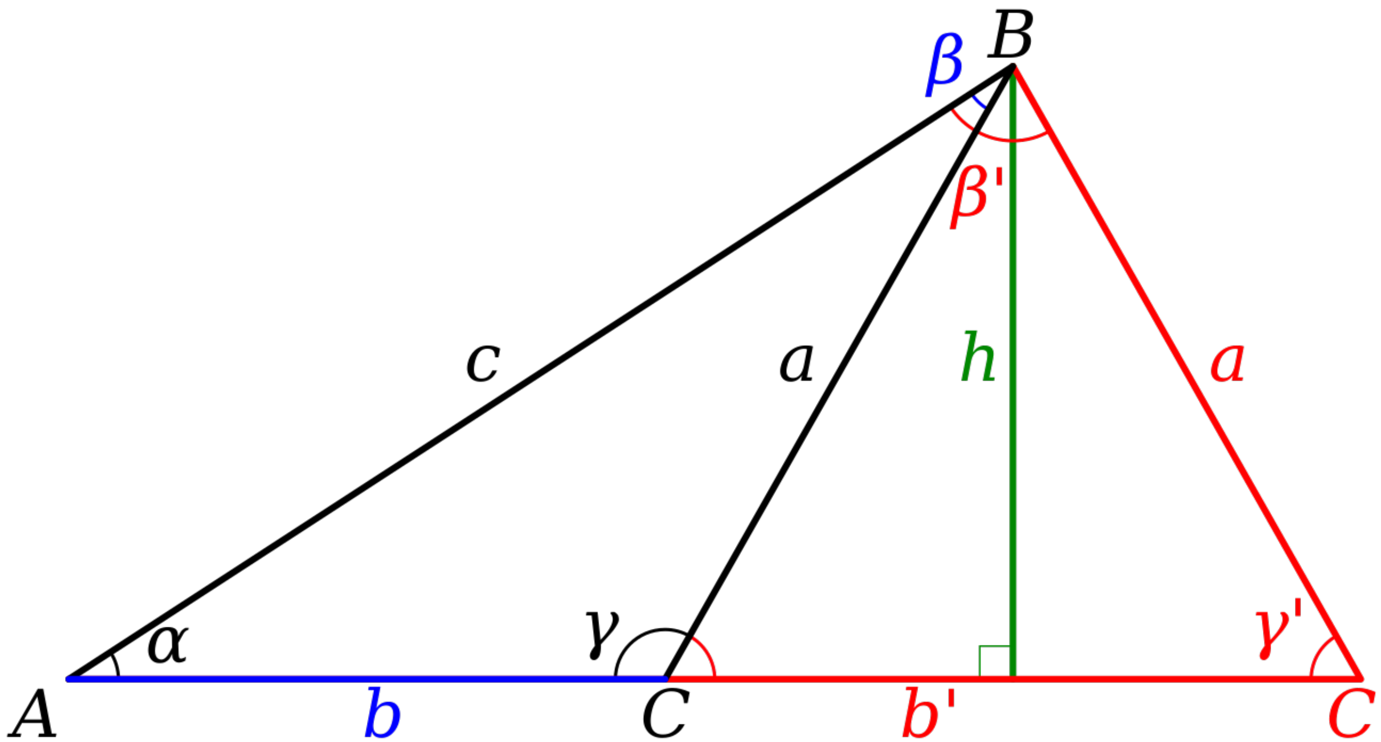
$$T = \frac{1}{2} b\left(c \sin \alpha\right) = \frac{1}{2} c\left(a \sin \beta\right) = \frac{1}{2} a\left(b \sin \gamma\right).$$

Multiplying these by $\frac{2}{abc}$ gives

$$\frac{2T}{abc} = \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

The ambiguous case of triangle solution

When using the law of sines to find a side of a triangle, an ambiguous case occurs when two separate triangles can be constructed from the data provided (i.e., there are two different possible solutions to the triangle). In the case shown below they are triangles ABC and ABC'.



Given a general triangle, the following conditions would need to be fulfilled for the case to be ambiguous:

- The only information known about the triangle is the angle α and the sides a and c.
- The angle α is acute (i.e., $\alpha < 90°$).
- The side a is shorter than the side c (i.e., $a < c$).
- The side a is longer than the altitude h from angle β, where $h = c \sin \alpha$ (i.e., $a > h$).

If all the above conditions are true, then each of angles β and β' produces a valid triangle, meaning that both of the following are true:

$$\gamma' = \arcsin \frac{c \sin \alpha}{a} \quad \text{or} \quad \gamma = \pi - \arcsin \frac{c \sin \alpha}{a}.$$