Black Holes, Problem Set 15 for Monday, Dec. 2

Reading from *Exploring Black Holes*

Finish Chapter B. You also have the reading Rebecca requested.

For Problem Set 15

Problem 1 — Entropy

On the first page of the reading from *Black Holes and Time Warps,* is a definition of entropy: "Entropy is the logarithm of the number of ways that all atoms and molecules in our chosen region can be distributed, without changing that region's macroscopic appearance." Let us consider flipping 20 coins at once. Let us consider the "macroscopic appearance" of those 20 coins to be the number that have heads up. The formula for probability of having n heads up out of a possible N coin flips is: $\binom{N}{n} 0.5^N$.

The thing in parenthesis is read aloud as "N choose n," and it is the number of independent ways of having n heads among N coin flips. The formula for N choose n is: $\binom{N}{n} = \frac{N!}{(N-n)!\,n!}$

(a) Evaluate $\binom{N}{n}$ with $N = 20$ and $n = 2$. Don't reach for your calculator. There are tremendous cancellations between the factorials that make this one quite easy to evaluate by hand.
(b) Evaluate $\binom{N}{n}$ with $N = 20$ and $n = 4$. You probably want a calculator now, but do your calculator a favor and do the cancellations first and then punch in what is left.
(c) So if the macroscopic appearance is 2 heads, take the natural logarithm of what you got in (a) to get the entropy of this situation. (The base doesn't really matter, but the natural log is more "natural" than $\log_{10}$ which we got into historical habit of using using simply because we have 10 fingers.)
(d) Also take the natural log of the answer in (b) to get the entropy of the situation with 4 heads.
(e) Finally, what is the entropy of the state with 0 heads? That's easy. There is only 1 way of rolling 0 heads, because each and every coin has to come up tails.

DISCUSSION: The core theoretical conundrum with dumping your garbage into a black hole is that once the garbage arrives at the singularity, the macroscopic state of the system is perfectly defined. It is a black hole with a mass, and any two black holes with the same mass have exactly the same state. It doesn't matter what kind of garbage you put in. It could have been the complete works of Shakespeare or it could have been the telephone book. So there is apparently only 1 final state despite the plethora of initial states that could have led to it.

You know the entropy when there is only one state corresponding to a macroscopic situation. It is the same as what you found in (e). So we started at a state with lots of entropy (all the kinds of garbage that can swirl around a black hole), and we finished with a state with entropy 0. This seems to violate the Second Law of Thermodynamics, which is the law of entropy increase. Hmmmm. pp. 422-448 are a long discussion of this theoretical conundrum. The conundrum has the name “The Black Hole Information Paradox.”

Problem 2 — Deriving the Rain Metric

Through various calculations in Chapter 3, we learned that however you toss a stone (or a photon) into a black hole, by the time the stone gets to the event horizon, it is going the speed of light in shell coordinates. And it has 0 speed in bookkeeper coordinates. This is not illuminating!

To understand the experience of something passing through the event horizon, we are forced to change coordinates. To understand the experience, we are going to change to the falling object’s coordinates. Isn’t that somewhat circular reasoning: to understand what is happening to some object, whose motion in the only coordinate systems we so far have we don’t understand, we are going to change to its coordinates!? Actually, we’ll keep using the Schwarzschild bookkeeper’s r, but we are going to use the time kept by the thing that is falling, ...

More specifically, imagine there is somebody called “The Rainmaker.” And this person has been releasing raindrops from a distance $r = \infty$ since the dawn of time, and now here we are in the present and these raindrops are at all stages of falling into the black hole. Some of the raindrops are still outside the event horizon, some are inside, and some are just crossing the event horizon at the speed of light (according to a shell observer). Of course, according to each of the raindrops, it is enjoying free fall and is certainly not moving at the speed of light. Each raindrop is comfortably at rest in its own frame and has been during its entire fall. Each raindrop is also wearing a wristwatch.

An important thing to note is that the raindrops that were released later have a larger time on their wristwatches when they cross the event horizon than the ones that were released earlier. (In fact, it took them an infinite amount of time to get from $r = \infty$ where they were released from rest to pick up speed and arrive near the black hole, but we are going to ignore that ∞ because you’ll see that we only need to concern ourselves with differences in the times on the raindrops’ wristwatches.)

So here is our situation: we are going to measure elapsed time between any two events by comparing the difference in time on the wristwatches of the two raindrops that happened to be at those two events. We will call that Δt_{rain}. And we are going to use Schwarzschild bookkeeper r to measure the difference in the two events’ radial positions. We’ll call that Δr. The question is what does the Schwarzschild metric look like in these coordinates!?

If you go back to p. 103 of *Spacetime Physics,* you find our old friend, the inverse Lorentz transformation:

$$
\begin{aligned}
t' &= -v_{\text{rel}}\gamma x + \gamma t \\
x' &= \gamma x - v_{\text{rel}}\gamma t \\
y' &= y \\
z' &= z
\end{aligned}
\qquad \text{(L-11a)}
$$

I'm going to drop the cumbersome "rel" subscript because there is only one velocity in what follows, and I am going to leave the *y* and z coordinates out because we are going to confine ourselves to motion along one direction which shortly become the radial direction. Finally, I am going to apply this formula to two points which have coordinates (t_1, x_1), and (t_2, x_2). Of course these points also have coordinates (t_1', x_1') and (t_2', x_2'), and Equations L-11a tell you how to get the primed (rocket) coordinates from the unprimed (lab) coordinates. So we have

$t_1' = -v\,\gamma\,x_1 + \gamma\,t_1$ (I'll call this equation *)
$x_1' = \gamma\,x_1 + -v\,\gamma\,t_1$ (And call this **)

and we have

$t_2' = -v\,\gamma\,x_2 + \gamma\,t_2$ (And this ***)
$x_2' = \gamma\,x_2 + -v\,\gamma\,t_2$ (And this ****)

It turns out we don't need Eq. ** and Eq. ****. Happy day. But we do need Eq. * and Eq. ***.

(a) Subtract Eq. * from Eq. *** and write down the resulting equation.

(b) Instead of using primed and unprimed to represent rocket and lab coordinates, make the following name changes:

$t' \to t_{\text{rain}}$
$x' \to r_{\text{rain}}$

$t \to t_{\text{shell}}$
$x \to r_{\text{shell}}$

Rewrite the Equation you found in (a) with these new names. You still have two events, so make sure you copy over all the 1's and 2's that are littering the equations.

(c) Now $t_{\text{rain},2} - t_{\text{rain},1}$ is of course what we call $\Delta\,t_{\text{rain}}$, and $r_{\text{rain},2} - r_{\text{rain},1}$ is what we call $\Delta\,r_{\text{rain}}$, and similarly for $\Delta\,t_{\text{shell}}$ and $\Delta\,r_{\text{shell}}$, so recopy the equation from (b) with those nice shorthands.

(d) Now we are transforming from shell coordinates to rain coordinates, and that depends on how fast the rain is shooting past the shell. But we have calculated that multiple times in Chapter 3. So put in that $v = -\sqrt{\frac{2M}{r}}$ is the v in question — it is the velocity of the raindrop (the rocket) shooting radially inward in shell coordinates (the lab). Also use this formula for v in the formula for γ. With these values for v and γ you can rewrite the equation in (c).

(e) Now we said we were going to use the bookkeeper Δr instead of shell's Δr_{shell}, and fortunately, that is something we have used many times too. The relation is $\Delta r_{\text{shell}} = \Delta r \big/ \sqrt{1 - 2M/r}$. Also, we are working our way to using Δt_{rain} and it will help us get there to get rid of Δt_{shell} at this step. So also use $\Delta t_{\text{shell}} = \Delta t \sqrt{1 - 2M/r}$.
(f) Next solve what you got in (e) for Δt.
(g) Now for the final grand step. Stick in what you got for Δt in (f) into the Schwarzschild metric and simplify (we will make heavy use of your result in Problem 3):

$$(\Delta\tau)^2 = (1 - 2M/r)(\Delta t)^2 - \frac{1}{1-2M/r}(\Delta r)^2 - r^2(\Delta\phi)^2$$

Problem 3 — The Path of Light in the Rain Metric

In Problem 2, you derived the rain metric. You should have obtained:

$$(\Delta\tau)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t_{\text{rain}})^2 - 2\sqrt{\frac{2M}{r}}\,\Delta t_{\text{rain}}\,\Delta r - (\Delta r)^2 - r^2(\Delta\phi)^2$$

For a particle going radially inward or outward $\Delta\phi = 0$, and this simplifies to:

$$(\Delta\tau)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t_{\text{rain}})^2 - 2\sqrt{\frac{2M}{r}}\,\Delta t_{\text{rain}}\,\Delta r - (\Delta r)^2$$

Now light is "lightlike," meaning that the only paths it can take have $\Delta\tau = 0$, so for light:

$$0 = \left(1 - \frac{2M}{r}\right)(\Delta t_{\text{rain}})^2 - 2\sqrt{\frac{2M}{r}}\,\Delta t_{\text{rain}}\,\Delta r - (\Delta r)^2$$

(a) Multiply that equation through by $-\frac{1}{2}\frac{1}{(\Delta t_{\text{rain}})^2}$. Then thinking of the resulting quadratic equation in the variable $\frac{\Delta r}{\Delta t_{\text{rain}}}$, with $a = \frac{1}{2}$, $b = \sqrt{\frac{2M}{r}}$, and $c = -\frac{1}{2}\left(1 - \frac{2M}{r}\right)$ do the usual $\left(\frac{\Delta r}{\Delta t_{\text{rain}}}\right)_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ business and write down two equations, one for $\left(\frac{\Delta r}{\Delta t_{\text{rain}}}\right)_{+}$ and one for $\left(\frac{\Delta r}{\Delta t_{\text{rain}}}\right)_{+}$.
(b) In (a) you got two equations. Put in an enormous value of r, like $r = 200\,M$, into the equation. What are your two answers for light shone outward and the light shone inward in these funny coordinates? Keep in mind these are rain coordinates. The only thing about these coordinates that is highly physical is that we are using the raindrops' wristwatch time. We are still using Schwarzschild bookkeeper r.
(c) Now let's see what we get inside the event horizon. Stick in a small value of r, like $r = \frac{1}{2}M$. What are $\left(\frac{\Delta r}{\Delta t_{\text{rain}}}\right)_{+}$ and $\left(\frac{\Delta r}{\Delta t_{\text{rain}}}\right)_{+}$. What does this tell you about light shone outward and inward, at least according to nice physical raindrop wristwatch time and Schwarzschild bookkeeper r?

Problem 4 — A Question About *Black Holes and Time Warps*

Name: ____________________

The discussion in *Black Holes and Time Warps* on pp. 422-448 of is advanced and speculative stuff, and I am not sure that we can do much with it. At least as recently as 3 years ago, Joe Polchinski was still giving lectures on what is termed “The Black Hole Information Paradox,” and he believes string theory sheds some light on the resolution. See https://youtu.be/2yx66ZEVavg. In any case, perhaps you all will come up with some questions that we can meaningfully probe: