Black Holes, Calculus Self-Assessment

Thursday, Sept. 29, 2024

1. Derivatives

This is an equation describing light falling into a rotating black hole:

$$\frac{dr}{dt} = -\frac{\sqrt{1-2/r}}{1+2/r}$$

In this equation, r is a function of time, $r(t)$, and r must always be positive.

(a) Assuming $r > 2$, and nothing else, what can you say about the right-hand side of this equation?

(b) Now that you have said that, using the whole equation, what can you say about $r(t)$ (again assuming $r > 2$, and nothing else).

(c) Suppose the equation was the vastly simpler equation to solve:

$$\frac{dr}{dt} = -3t^2.$$

What $r(t)$ solves this equation?

(d) Don't forget the constant term in your answer to (c) (there should be an unknown constant in your answer).

(e) If the unknown constant is 10, at what time does $r(t) = 2$?

See reverse.

2. Integrals

Here is a screwball metric that I just made up to have something:

$d\,s^2 = d\,x^2 + y^2\,d\,y^2$

This should not mean *anything* to you yet. If an object moves in the y direction only, this equation simplifies to

$d\,s = y\,d\,y$

If the object moves from $y = y_1$ to $y = y_2$, then the total distance moved is

$s = \int_{y_1}^{y_2} y\ d\,y$

(a) Graph the function you are integrating from $y_1 = 1$ to $y_2 = 2$.

(b) Just looking at your graph, estimate the integral.

(c) Using standard integral techniques, what is the exact value for the integral?