# Cosmology

The first half of what follows is motivation, not derivation. Robertson and Walker proved (in 1929 and 1936) that the most general homogeneous isotropic spacetime is that which had already been given by Friedman in 1924. Once you have been through the motivation, you need to free yourself from it, because we are interested in starting with the most general solution, and narrowing down with observations to the particular solution of Einstein's equations we actually live in.

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## Newtonian Big Bang Cosmology

Imagine you can see a galaxy a distance R from you, and you consider yourself to be at the center of a uniform Newtonian space filled with dust of density $\rho$. The galaxy is moving away from you with speed $\frac{dR}{dt}$. You can imagine that Newton's Universal law of Gravitation applies to you and the galaxy in the following way:

A sphere of radius $R$ contains a mass $M = \frac{4}{3}\pi R^3 \rho$ of dust. Now dust is not created or being destroyed, so you can see that the if the galaxy is moving away from you at speed $\frac{dR}{dt}$ and the dust at the edge of the sphere is moving away at the same speed, then it must be that $\rho$ decreases as $R$ increases, and so $\rho = \frac{M}{\frac{4}{3}\pi R^3}$.

The speed of the galaxy should be decreasing and its decrease is due to the Newton's Universal Law of Gravitation, so

$$m_{\text{galaxy}} \frac{d^2 R}{d t^2} = -\frac{G M m_{\text{galaxy}}}{R^2}$$

In this course, we usually set $G = 1$, and also we can cancel the mass of the galaxy off of both sides, leaving

$$\frac{d^2 R}{d t^2} = -\frac{M}{R^2}$$

To integrate this equation, multiply both sides by $\frac{dR}{dt}$.

$$\frac{dR}{dt}\frac{d^2 R}{d t^2} = -\frac{M}{R^2}\frac{dR}{dt}$$

and recognize that the left hand side is

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dR}{dt}\right)^2$$

and the right hand side is

$$\frac{d}{dt}\frac{M}{R}$$

That means that they are equal up to a constant, which I will call $-b$.

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{M}{R} - b \quad \text{or} \quad \left(\frac{dR}{dt}\right)^2 = \frac{2M}{R} - 2b$$

This you will see is a lot like Einstein’s cosmology. It will turn out that $b = \frac{1}{2}$ for a big bang with enough mass in the volume $R$ to eventually bring the expansion to a halt. That isn’t entirely satisfying to me, and it should be to you either, but it is a good whack of motivation for what follows.

# A Homogeneous and Isotropic Universe

# The Effective Potential with Only Dust

The following must be satisfied for Einstein’s equations containing uniform dust:

$$V_{\text{eff}}(R) = -\frac{2M}{R} \qquad \text{and} \qquad -1 = \left(\frac{dR}{dt}\right)^2 + V_{\text{eff}}(R)$$

Then (focusing on the positive square root):

$$\frac{dR}{dt} = \sqrt{-1 + \frac{2M}{R}}$$

A. If $R = \frac{M}{5}$, what is must $\frac{dR}{dt}$ be? $\frac{dR}{dt} = 3$

B. If $R = 2M/5$ what must $\frac{dR}{dt}$ be? $\frac{dR}{dt} = 2$

C. If $R = 2M$ what must $\frac{dR}{dt}$ be? $\frac{dR}{dt} = 0$

COMMENT: Ah-hah we have found the turning point of the universe. When it grows to a radius $R = 2M$ it stops expanding and after that $\frac{dR}{dt}$ is negative.

Let’s focus on the negative square root and look at some tiny values of $R$. Let’s make the values so small that the -1 in $\sqrt{-1+\frac{2M}{R}}$ is negligible compared to $\frac{2M}{R}$.

D. Put in $R=\frac{M}{5000}$. Then $\frac{dR}{dt}=-\sqrt{-1+\frac{2M}{R}}\approx-\sqrt{\frac{2M}{M/5000}}=-\sqrt{10\,000}=-100$.

Or put in $R=\frac{M}{20\,000}$. Then $\frac{dR}{dt}=\sqrt{\frac{2M}{M/20\,000}}=\sqrt{40\,000}=-200$.

COMMENT: What you are approaching in D is called "the big crunch," and it is a possible endpoint for the universe if there were enough mass in the universe to slow the expansion to a halt. All evidence is actually that there is not enough mass to slow the expansion, and the universe will expand forever.

# Introducing Variables Referencing the State of the Universe Today

Let's go back to

$$1=\left(\frac{dR}{dt}\right)^2+V_{\text{eff}}(R)=\left(\frac{dR}{dt}\right)^2-\frac{2M}{R}$$

We define $R(t)=R_0\,a(t)$

Then

$$1=R_0{}^2\left(\frac{da}{dt}\right)^2-\frac{2M}{R_0\,a}$$

or

$$\frac{1}{R_0{}^2}=\left(\frac{da}{dt}\right)^2-\frac{2M}{R_0{}^3\,a}$$

Finally define the density of dust today

$$\rho_0=\frac{M}{\frac{4}{3}\pi R_0{}^3} \quad \text{or } R_0{}^3=\frac{M}{\frac{4}{3}\pi\rho_0}=$$

Then

$$\frac{1}{R_0{}^2}=\left(\frac{da}{dt}\right)^2-\frac{2M}{\frac{M}{\frac{4}{3}\pi\rho_0}a}=\left(\frac{da}{dt}\right)^2-\frac{\frac{8\pi}{3}\rho_0}{a}$$

# Adding “Dark Energy”

$$\frac{1}{R_0{}^2} = \left(\frac{da}{dt}\right)^2 - \frac{2\,M}{\frac{M}{\frac{4}{3}\,\pi\,\rho_0}\,a} = \left(\frac{da}{dt}\right)^2 - \frac{\frac{8\,\pi}{3}\,\rho_0}{a}$$