Classical Mechanics — Exam 1

Monday, Sept. 29, 2025 — Covering *Six Ideas,* Volume C, Chapters C1-C7

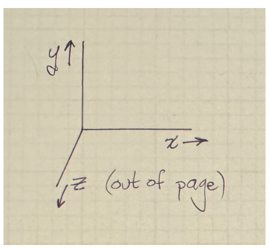
0. Vector Garbage (2 pts)

Cross out the statements that are garbage:

(a) $|\vec{v}| = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

(b) $|\vec{v}| = -5$ meters / second

(c) $\hat{e}_y \times \hat{e}_x = \hat{e}_z$ (in a usual right-handed coordinate system like the one drawn below)



(d) $\vec{L} = I\,|\vec{\omega}|$

1. A Collision Problem Requiring Conservation of Momentum (4 pts)

Conservation of momentum for an isolated system of N particles says:

$$\vec{p}_{1f} + \vec{p}_{2f} + \ldots + \vec{p}_{Nf} = \vec{p}_{1i} + \vec{p}_{2i} + \ldots + \vec{p}_{Ni} \qquad \text{(\textit{Six Ideas,} Eq. C5.1)}$$

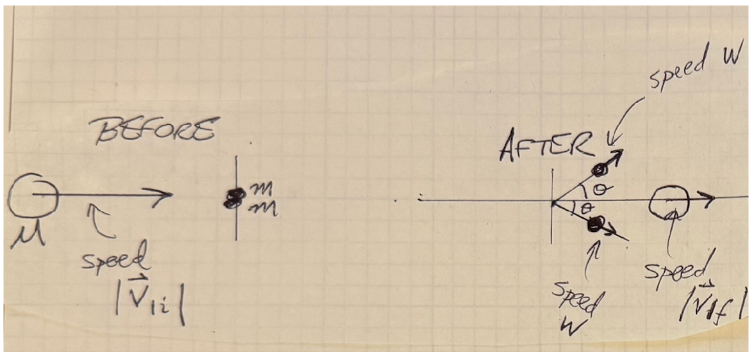
You have never used this equation with more than two incoming and outgoing particles, but in this problem we'll use it with three particles just to have something new to do.

Assume that a heavy object with mass M comes in from the left and is going to the right with speed $|\vec{v}_{1i}|$. It's going to collide with two lighter particles at rest, and we'll finally get to those two particles in part (c).

Assume the usual coordinate system with the x coordinate to the right and the y coordinate up. We can ignore the z coordinate entirely in this problem.

(a) In the usual $\begin{pmatrix} p_x \\ p_y \end{pmatrix}$ component style write down the initial momentum, $\vec{p}_{1i}$, in terms of M and $|\vec{v}_{1i}|$.

(b) The heavy object collides with the two lighter particles at the origin, and the heavy object leaves the collision still going to the right, but now with the reduced speed $|\vec{v}_{1f}|$. Still using the usual component style, write down the final momentum of the heavy object, $\vec{p}_{1f}$, in terms of M and $|\vec{v}_{1f}|$, and also write down the change in momentum of the heavy object $\vec{p}_{1f} - \vec{p}_{1i}$.



(c) The two identical, lighter objects leave the origin one having angle θ above the x-axis and one having angle θ below the x-axis and both having the same speed which we'll call w (no vector or magnitude symbol around w — it is just a speed).

The two lighter objects have mass m. Let's number these as the 2nd and 3rd particles. Write down their momenta $\vec{p}_{2f}$ and $\vec{p}_{3f}$, again using the usual component style, in terms of m, w, $\cos\theta$, and $\sin\theta$.

(d) Compute the sum $\vec{p}_{2f} + \vec{p}_{3f}$, again using the usual component style.

(e) The change in momentum you found in (b) must be ***opposite*** the momentum you summed up in (d) for momentum to be conserved. Put that fact into an equation, still using the usual component style. HINT: The y component of the equation will be trivial if you did things right (0 = 0).

(f) The x component of the equation can be solved for w if everything else is known. Finish this problem by doing that.

2. A Derivation about the Motion of the Center of Mass (4 pts)

The following equation ***defines*** the center of mass for a system of N particles:

$\vec{r}_{\text{CM}} \equiv \frac{1}{M}\left(m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots + m_N \vec{r}_N\right)$ (*Six Ideas,* Eq. C4.1)

(a) Take one derivative with respect to time of this equation. When you take the time derivative, you can assume $m_1, m_2, \ldots, m_N$ and also $M \equiv m_1 + m_2 + \ldots + m_N$ are constants. (I mention the constancy of particle masses because for rockets shooting out propellant, coping with time-varying masses might make this more complicated.) After taking the time derivative, simplify by using $\vec{v}_1 \equiv \frac{d\vec{r}_1}{dt}$, $\vec{v}_2 \equiv \frac{d\vec{r}_2}{dt}$, ..., $\vec{v}_N \equiv \frac{d\vec{r}_N}{dt}$. Also use $\vec{v}_{\text{CM}} \equiv \frac{d\vec{r}_{\text{CM}}}{dt}$.

(b) Now use the Newtonian formulas, $\vec{p}_1 = m_1 \vec{v}_1$, $\vec{p}_2 = m_2 \vec{v}_2$, ..., $\vec{p}_N = m_N \vec{v}_N$ to make your formula exceedingly tidy. COMMENT: This derivation is quite easy to make relativistic, but we'll leave that complication out during the exam.

(c) The exceedingly tidy equation you wrote down in (b) is true before and after any amount of interactions have occurred among the particles. So that means you can use it to write down two formulas: a formula for $\vec{v}_{\text{CM}\,i}$ in terms of $\vec{p}_{1\,i}$, $\vec{p}_{2\,i}$, ..., $\vec{p}_{N\,i}$, and a formula $\vec{v}_{\text{CM}\,f}$ in terms of $\vec{p}_{1\,f}$, $\vec{p}_{2\,f}$, ..., $\vec{p}_{N\,f}$. Write down those two formulas.

(d) Consulting Eq. C5.1 reproduced at the beginning of Problem 1, write down a remarkably simple relationship between $\vec{v}_{\text{CM}\,i}$ and $\vec{v}_{\text{CM}\,f}$.

3. Moment of Inertia of a Rigid Body (5 pts)

Imagine a rod containing 20 particles. The particles are at d, $2d$, $3d$, ..., $10d$ to the right of center and the same amounts to the left of center. There is no particle at the origin. Somehow (spiderwebs?) the 20 points are held rigidly in this formation and we want to compute the moment of inertia if it is spun around the middle.

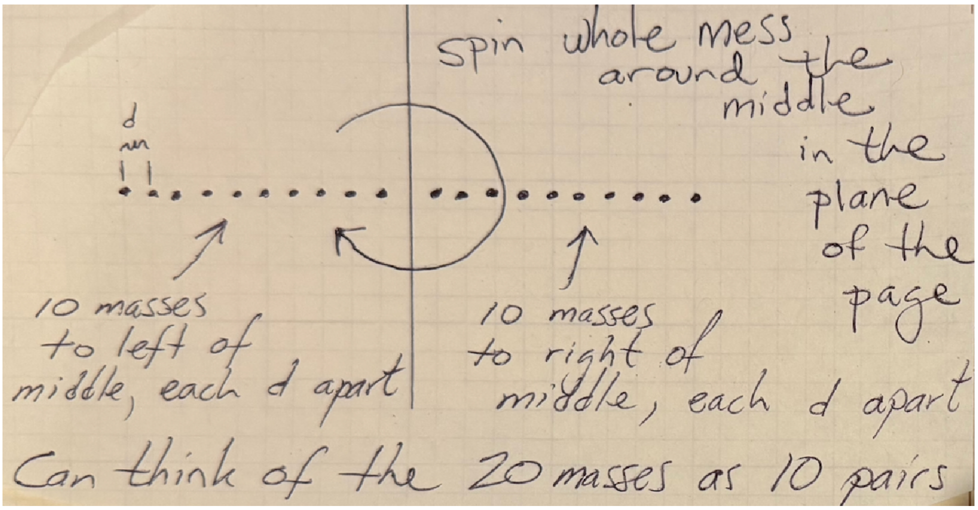


Figure C6.7 was a table of a bunch of α values for various rigid objects. It included a thin rod spinning around its midpoint, and for this situation the figure said that $\alpha = \frac{1}{3}$ in the rigid body rotation formula:

$$|\vec{L}| = I\,|\vec{\omega}| = \alpha M R^2\,|\vec{\omega}|$$

Of course, 20 particles is not the same as a continuous rod, it is just an approximation. So we won't expect to get an α exactly equal to $\frac{1}{3}$.

(a) Eq. C6.9, applied to our 10 pairs of particles says:

$$I = 2 \sum_{j=1}^{10} m_j r_j^2$$

Assuming all 20 of the particles add up to a total mass M, and all the m_j are identical, what is the relation between m_j and M? Put that relation into the formula for I, and pull the constants out in front of the sum to simplify.

(b) Assuming that $10d = R$, and also that $r_j = j \cdot d$, put those two facts into the sum and again pull constants out in front to simplify.

(c) Use $\sum_{j=1}^{10} j^2 = \frac{10\cdot 11\cdot 21}{6} = 385$ to simplify and then write your answer as a decimal. You'll get something noticeably larger than $\frac{1}{3}$, but not way larger.

(d) We can improve our approximation of a continuous rod as a bunch of points by imagining it is 200 particles which we can think of as 100 pairs. You don't have to repeat the work above. Just write down the answer and use that $\sum_{j=1}^{100} j^2 = \frac{100 \cdot 101 \cdot 201}{6} = 338\,350$ to write down a decimal expression in this improved case. You should get something quite a bit closer to 0.333333 than we got in (c).

(e) We can take the limit that the number of particles is infinite and then the sum over j of j^2 becomes identical to a Riemann sum. The Riemann sum is the Riemann sum for the integral:

$$\frac{M}{R} \int_0^R r^2 \, dr$$

What is this integral?

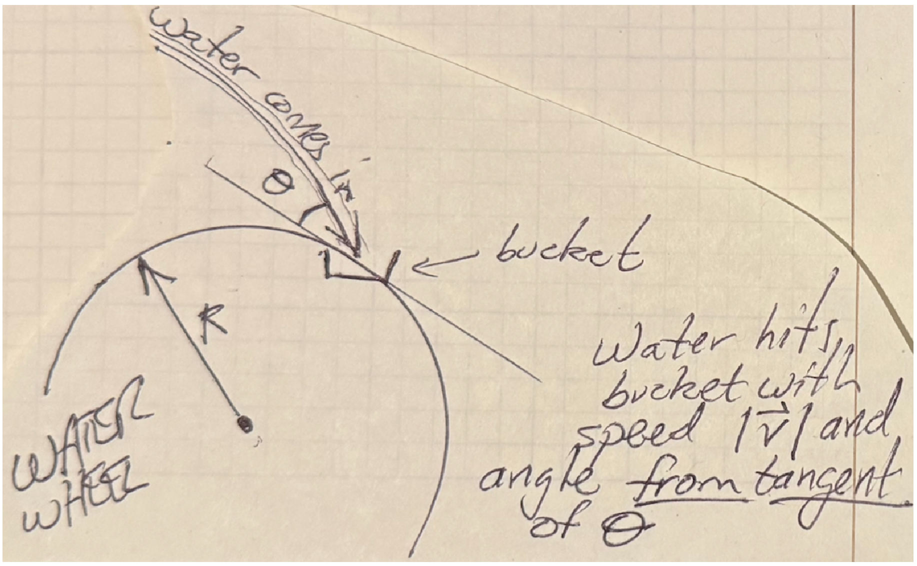
COMMENT: Now you have the short and sweet calculus-based derivation from which the value of α in Figure C6.7 for the rod came.

4. Angular Momentum Transfer (Twirl) (5 pts)

Conservation of angular momentum for an isolated system of N particles is

$$\vec{L}_{1f} + \vec{L}_{2f} + \ldots + \vec{L}_{Nf} = \vec{L}_{1i} + \vec{L}_{2i} + \ldots + \vec{L}_{Ni}$$

Let’s use this equation for something it wasn’t quite meant for: a water wheel. Buckets of water come in and are dumped, so it isn’t an isolated system, but we can still compute the twirl.



(a) Let’s assume that each bucket of water has a mass M of water in it, and the water enters the bucket with angle θ and speed $|\vec{v}|$ as drawn.

Using the fundamental formula $\vec{L} = \vec{r} \times \vec{p}$ what was the magnitude $|\vec{L}_i|$ of the angular momentum of the bucketful of water just before it got caught in the bucket?

HINT: Consult the drawing and be careful with angles. If your answer has $\sin\theta$ in it, put in $\theta = 0$ to see that $\sin\theta$ cannot be right. It might help to draw the $\vec{r}$ and $\vec{p}$ of the water at the point of impact onto my diagram.

(b) The rushing water greatly slows upon entering the slow-moving bucket. If the water wheel has angular velocity ω, what is this greatly-slowed ***speed*** (in terms of R and ω)?

(c) What is the magnitude $|\vec{L}_f|$ of the angular momentum of the greatly slowed bucketful of water in terms of M, R, and ω?

(d) OOPS, there was no part (d) ;P

(e) Using the right-hand rule, what direction do both $\vec{L}_f$ and $\vec{L}_i$ point? (I'm asking for an answer like "to the left," "to the right," "into the page," "out of the page," or something similarly descriptive.)

(f) Keeping in mind that the magnitude $|\vec{L}_f|$ is smaller than the magnitude $|\vec{L}_i|$ what is the direction of $\vec{L}_f - \vec{L}_i$? (As in Part (e), I'm asking for an answer like "to the left," "to the right," "into the page," "out of the page," or something similarly descriptive.)

(g) What is the magnitude $\left|\vec{L}_f - \vec{L}_i\right|$? (Magnitudes are positive!)

(h) Parts (f) and (g) together described the change in angular momentum of the water, and angular momentum is conserved, so the water wheel must have ***precisely the opposite change*** in its angular momentum. What is the direction of the water wheel's angular momentum change?

HINT/CROSS-CHECK: Your answer to (h) better comport with the right-hand rule applied to the movement of the water wheel.

COMMENTS:

In real life, the water wheel is used to grind grain, and the grinding apparatus is torquing the wheel in the opposite direction that the water is, so the water wheel doesn't actually gain or lose angular momentum. It just turns steadily.

The amount of angular momentum transfer ("twirl") found in (g) and (h) happens every time a bucket is filled. If the buckets are filled Δt apart, the average torque due to the water hitting the wheel would be $\vec{\tau} = \frac{\vec{L}_i - \vec{L}_f}{\Delta t}$.

Name ____________________

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2. / 4
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GRAND TOTAL

/ 20