ODE Assignment 1

for Monday, May 9, 2022

Problem 1 Logan p. 16 #4

Let $x_1(t) = e^{-t}\cos t$

Then $x_1'(t) = -e^{-t}\cos t - e^{-t}\sin t$

and $x_1''(t) = e^{-t}\cos t + e^{-t}\sin t + e^{-t}\sin t - e^{-t}\cos t$

$$x_1''(t) + 2x_1'(t) + 2x_1(t)$$
$$= \cancel{e^{-t}\cos t} + \cancel{e^{-t}\sin t} + \cancel{e^{-t}\sin t} - \cancel{e^{-t}\cos t}$$
$$+ 2(\cancel{-e^{-t}\cos t} \cancel{- e^{-t}\sin t}) + \cancel{2e^{-t}\cos t}$$
$$= 0$$

Let $x_2(t) = e^{-t}\sin t$

Then $x_2'(t) = -e^{-t}\sin t + e^{-t}\cos t$

and $x_2''(t) = e^{-t}\sin t - e^{-t}\cos t - e^{-t}\cos t - e^{-t}\sin t$

$$x_2''(t) + 2x_2'(t) + 2x_2(t)$$
$$= \cancel{e^{-t}\sin t} - \cancel{e^{-t}\cos t} - \cancel{e^{-t}\cos t} - \cancel{e^{-t}\sin t}$$
$$+ 2(\cancel{-e^{-t}\sin t} + \cancel{e^{-t}\cos t}) + \cancel{2e^{-t}\sin t}$$
$$= 0$$

Problem 1 Logan p. 10 #4 (CONT'D)

If you multiply $x_1(t) = e^{-t}\cos t$ by A, then absolutely every term in the calculation of $x_1''(t) + 2x_1'(t) + 2x_1(t)$ gets multiplied by A. $A \cdot 0 = 0$. Similarly if you multiply $x_2(t) = e^{-t}\sin t$ by B, then absolutely every term in the calculation of $x_2''(t) + 2x_2'(t) + 2x_2(t)$ gets multiplied by B. $B \cdot 0 = 0$.

If you take the combination

$Ax_1(t) + Bx_2(t)$ and compute

$$\frac{d^2}{dt^2}\left(Ax_1(t) + Bx_2(t)\right) + 2\frac{d}{dt}\left(Ax_1(t) + Bx_2(t)\right) + 2\left(Ax_1(t) + Bx_2(t)\right)$$

$$= A\left(x_1''(t) + 2x_1'(t) + 2x_1(t)\right) + B\left(x_2''(t) + 2x_2'(t) + 2x_2(t)\right)$$

$$= A \cdot 0 + B \cdot 0 = 0$$

Problem 2 Logan p. 10 #6

Consider the differential equation

$$x'(t) + 2x(t) = t^2 + 4t + 7$$

Try $x(t)$ of the form $at^2 + bt + c$.
What conditions are there on a, b, and c?

$$x'(t) = 2at + b$$

$$x'(t) + 2x(t) = 2at + b + 2(at^2 + bt + c)$$

This is supposed to equal $t^2 + 4t + 7$

This requires $2a = 1$ ← from coefficient of t^2

$2a + 2b = 4$ ← from coefficient of t

$b + 2c = 7$ ← from constant term

$$a = \frac{1}{2}, \quad b = \frac{3}{2}, \quad c = \frac{11}{4}$$

Icchy numbers. Double-check:

$$\frac{d}{dt}\left(\frac{1}{2}t^2 + \frac{3}{2}t + \frac{11}{4}\right) = t + \frac{3}{2}$$

$$x'(t) + 2x(t) = t + \frac{3}{2} + t^2 + 3t + \frac{11}{2}$$

$$= t^2 + 4t + 7 \quad ✓✓$$

Problem 3 Logan p. 10 #8

Consider the differential equation

$$t^2 x''(t) - 6x(t) = 0$$

Try $x(t) = t^m$ in this equation.

$$x''(t) = m(m-1)t^{m-2}$$

$$t^2 x''(t) - 6x(t) = t^2 m(m-1)t^{m-2} - 6t^m$$

$$= t^m(m(m-1) - 6)$$

Only works if $m(m-1) = 6$

$m = 3$ is a solution (because $3 \cdot 2 = 6$)

$m = -2$ is also a solution (because $(-2)(-3) = 6$)

So t^3 and t^{-2} are two solutions. Or any linear combination

$$At^3 + Bt^{-2}$$

Problem 4 Logan p. 15 #2

First we tabulate x^2+t^2 for a buench of values of (t, x)

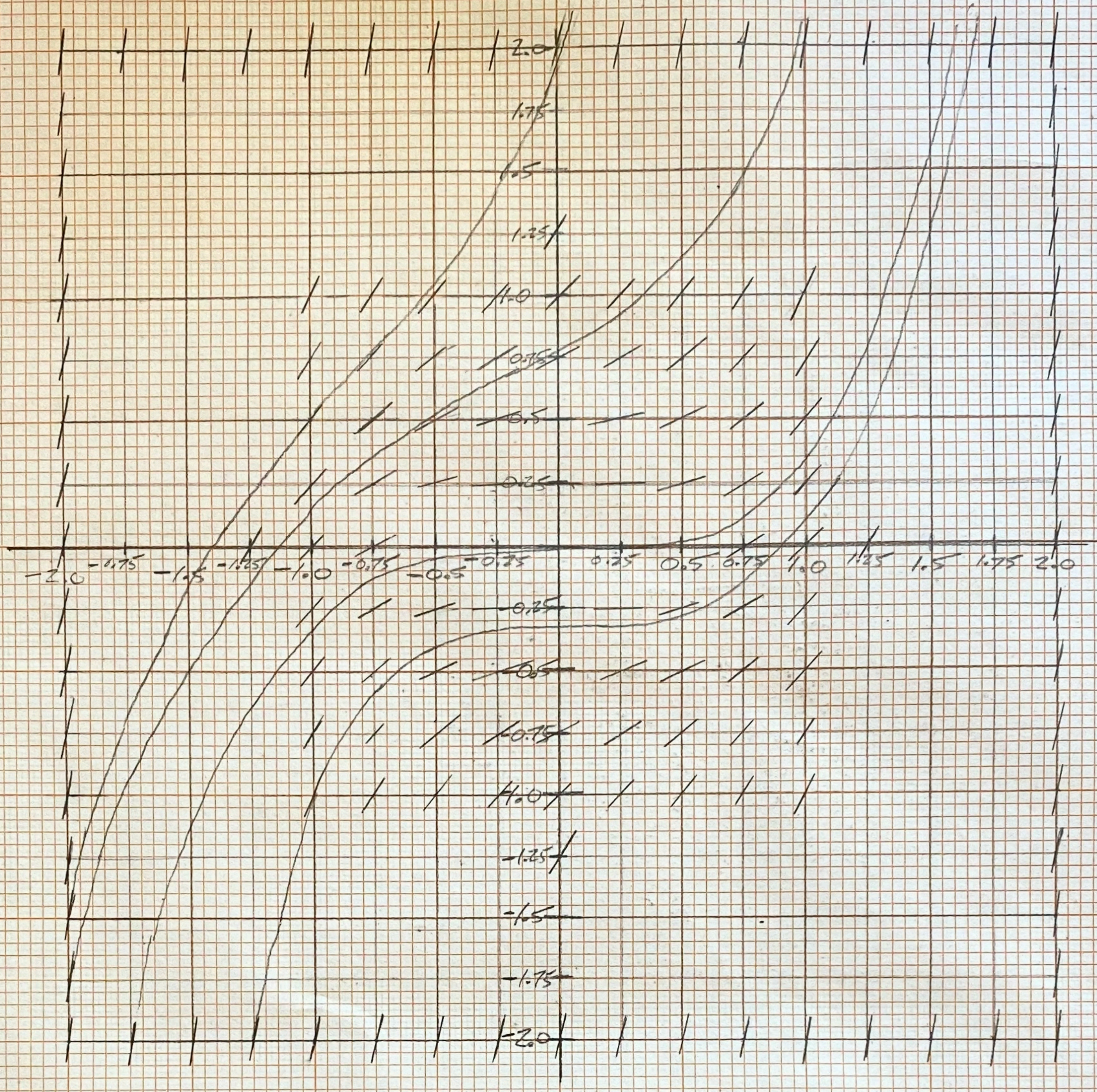
x^2+t^2 $x \rightarrow$

	-2.0	-1.75	-1.5	-1.25	-1.00	-0.75	-0.5	-0.25	0	
-2.0	8.0	7.06	6.25	5.56	5.0	4.56	4.25	4.06	4.0	
-1.75	7.06	6.12	5.31	4.62	4.06	3.62	3.31	3.12	3.06	
-1.5	6.25	5.31	4.5	3.81	3.25	2.81	2.5	2.31	2.25	
-1.25	5.56	4.62	3.81	3.12	2.56	2.12	1.81	1.62	1.56	...
-1.0	5.0	4.06	3.25	2.56	2.0	1.56	1.25	1.06	1.0	
-0.75	4.56	3.62	2.81	2.12	1.56	1.12	0.81	0.62	0.56	
-0.5	4.25	3.31	2.5	1.81	1.25	0.81	0.5	0.31	0.25	
-0.25	4.06	3.12	2.31	1.62	1.06	0.62	0.31	0.12	0.06	
0.0	4.0	3.06	2.25	1.56	1.0	0.56	0.25	0.06	0.0	

$\downarrow t$

$\vdots$

This table has a lot of symmetries. It is not necessary to do the positive values of x and t to make a nice graph.


2.0
1.75
1.5
1.25
1.0
0.75
0.5
0.25
-0.25
-0.5
-0.75
-1.0
-1.25
-1.5
-1.75
-2.0
-2.0
-1.75
-1.5
-1.25
-1.0
-0.75
-0.5
-0.25
0.25
0.5
0.75
1.0
1.25
1.5
1.75
2.0

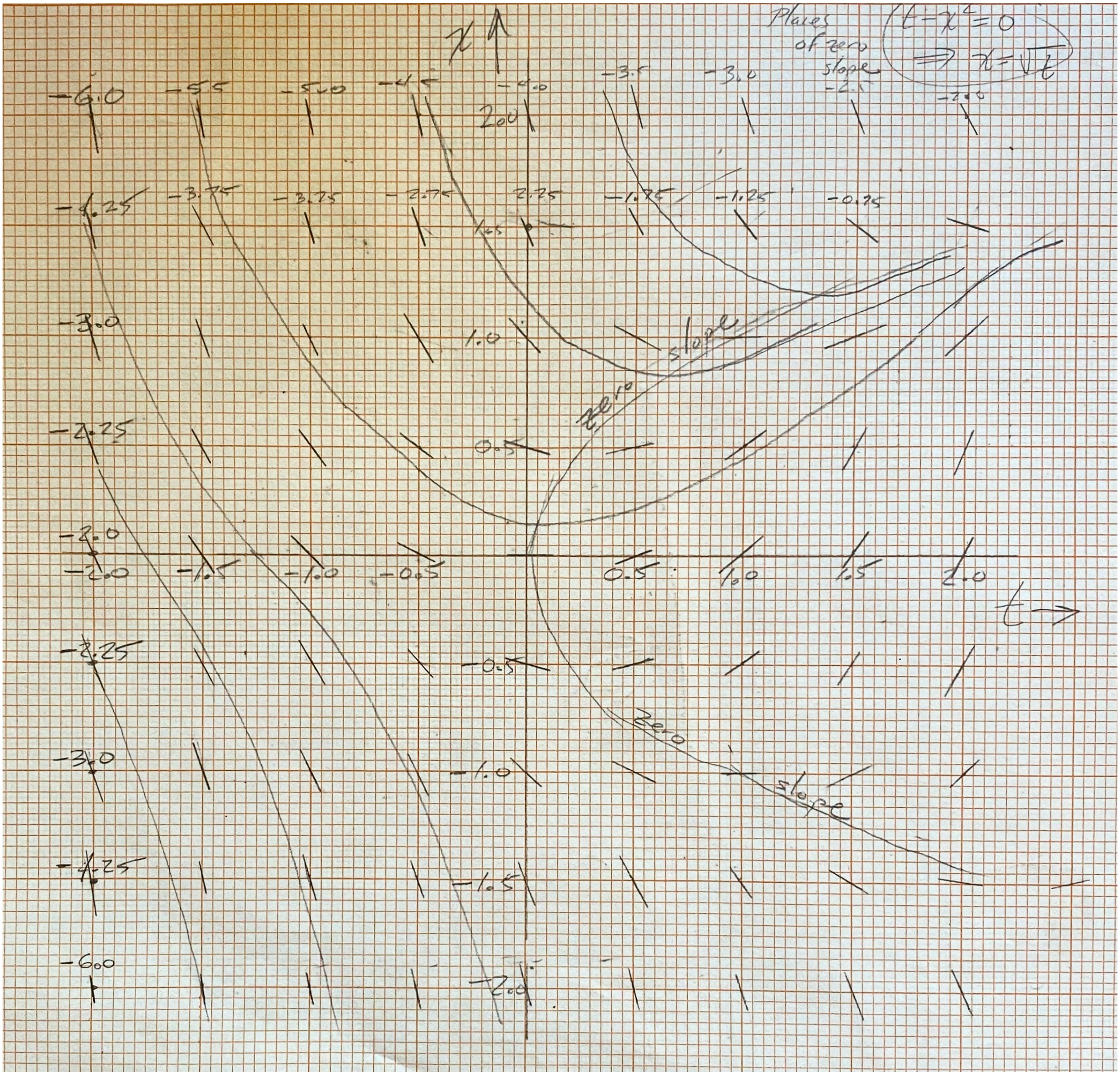
Problem 5. Logan p. 15 #4

First we tabulate $t-x^2$ for a bunch of values of (t,x)

$x \longrightarrow$

$\uparrow t$

$t-x^2$	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0
2.0	-2.0	-0.25	1.0	1.75	2.0	1.75	1.0	-0.25	-2.0
1.5	-2.5	-0.75	0.5	1.25	1.5	1.25	0.5	-0.75	-2.5
1.0	-3.0	-1.25	0.0	0.75	1.0	0.75	0.0	-1.25	-3.0
0.5	-3.5	-1.75	-0.5	0.25	0.5	0.25	-0.5	-1.75	-3.5
0.0	-4.0	-2.25	-1.0	-0.25	0.0	-0.25	-1.0	-2.25	-4.0
-0.5	-4.5	-2.25	-1.5	-0.75	-0.5	-0.75	-1.5	-2.75	-4.5
-1.0	-5.0	-3.25	-2.0	-1.25	-1.0	-1.25	-2.0	-3.25	-5.0
-1.5	-5.5	-3.75	-2.5	-1.75	-1.5	-1.75	-2.5	-3.75	-5.5
-2.0	-6.0	-4.25	-3.0	-2.25	-2.0	-2.25	-3.0	-4.25	-6.0



Places of zero slope

$t - x^2 = 0$

$\Rightarrow x = \sqrt{t}$