ODE Assignment 2

Due end-of-day Tuesday May 10, 2022

Problem 1

Logan p. 19 #2

Solve the initial-value problem:

$$x'(t) = \frac{t+1}{\sqrt{t}} \qquad x(1) = 4$$

$$\boxed{x(t)} = \int_1^t \frac{s+1}{\sqrt{s}}\, ds + 4 = \int_1^t \left(s^{1/2} + s^{-1/2}\right) ds + 4$$

$$= \left(\frac{s^{3/2}}{3/2} + \frac{s^{1/2}}{1/2}\right)\Bigg|_1^t + 4$$

$$= \frac{2}{3} t^{3/2} + 2t^{1/2} - \frac{2}{3} - 2 + 4$$

$$\boxed{= \frac{2}{3} t^{3/2} + 2t^{1/2} + \frac{4}{3}}$$

Plug in $t=1$ to double-check $\frac{2}{3} + 2 + \frac{4}{3} = 4$ ✓✓

Take derivative to double-check

$$\cancel{\frac{2}{3}} \cdot \cancel{\frac{3}{2}}\, t^{1/2} + \cancel{2} \cdot \frac{1}{\cancel{2}}\, t^{-1/2} = t^{1/2} + t^{-1/2}$$

$$= \frac{t+1}{\sqrt{t}} \quad ✓✓$$

Problem 2 Logan p. 20 #4

4(a) $x' = te^{-2t}$

Guess $x(t) = (at^2+bt+c)e^{-2t} + d$

$$x'(t) = (2at+b)e^{-2t} - 2(at^2+bt+c)e^{-2t}$$
$$\stackrel{!}{=} te^{-2t}$$

Looks like $a=0$ $\quad -2b=1 \quad b-2c=0$

$b = -\frac{1}{2}$ $\quad c = \frac{1}{4}$

Solution is

$$\boxed{x(t) = \left(-\frac{1}{2}t + \frac{1}{4}\right)e^{-2t} + d}$$

d can be anything

Double-check by taking derivative

$$x'(t) = -\frac{1}{2}e^{-2t} - \left(\frac{1}{2}t + \frac{1}{4}\right)(-2)e^{-2t}$$
$$= \left(-\frac{1}{2} + t + \frac{1}{2}\right)e^{-2t} = te^{-2t} \checkmark\checkmark$$

Problem 2 (CONT'D)

4(b) $x'(t) = \frac{1}{t \ln t}$

$$x(t) = \int_a^t \frac{1}{s \ln s}\, ds + x(a)$$

$$\frac{ds}{s} = d(\ln s)$$

so let $r = \ln s$ $\quad dr = \frac{ds}{s}$

$$\boxed{x(t)} = \int_{\ln a}^{\ln t} \frac{dr}{r} + x(a)$$

$$= \ln \ln t - \ln \ln a + x(a)$$

$$\boxed{= \ln \frac{\ln t}{\ln a} + x(a)}$$

Double-check by differentiating:

$$\frac{d}{dt} \ln \frac{\ln t}{\ln a} = \frac{\cancel{\ln a}}{\ln t} \frac{1}{\cancel{\ln a}} \frac{1}{t} \quad \checkmark\checkmark$$

Problem 2 (CONT'D AGAIN)

4(c) $\sqrt{t}\,\dfrac{dx}{dt} = \cos\sqrt{t}$

$$x(t) = \int_a^t \frac{\cos\sqrt{s}}{\sqrt{s}}\,ds + x(a)$$

The obvious change of variables is $w = \sqrt{s}$

$$dw = \frac{1}{2}\frac{1}{\sqrt{s}}\,ds \qquad \text{or} \qquad \frac{ds}{\sqrt{s}} = 2\,dw$$

$$\boxed{x(t)} = \int_{\sqrt{a}}^{\sqrt{t}} \cos w \cdot 2\,dw = 2\sin w\Big|_{\sqrt{a}}^{\sqrt{t}}$$

$$\boxed{= 2\sin\sqrt{t} - 2\sin\sqrt{a}}$$

Let's double-check by taking the derivative:

$$\frac{dx(t)}{dt} = \cancel{2}\cos\sqrt{t} \cdot \frac{1}{\cancel{2}}\frac{1}{\sqrt{t}} \quad ✓✓$$

Problem 3 Logan p. 27 #8

$\frac{dx}{dt} = t^2 e^{-x}$ $(x(0) = \ln 2)$

$\int e^x dx = \int t^2 dt$ $e^x = \frac{t^3}{3} + c$ $c = 2$

$x = \ln\left(\frac{t^3}{3} + 2\right)$

Interval of existence is $\frac{t^3}{3} + 2 > 0$

$t^3 > -6$ $t > -\sqrt[3]{6}$

Double-check by taking derivative

$\frac{dx}{dt} = \frac{1}{\frac{t^3}{3} + 2} \frac{\cancel{3}t^2}{\cancel{3}} = t^2 e^{-x}$ ✓✓

Problem 4 Logan p.32 #2

$$T(t) = T_e + (T_0 - T_e)e^{-ht}$$

$t=0, \quad T_0 = 46°c$

$t_1 = 10\,min, \quad T_1 = 39°c$

$t_2 = 20\,min, \quad T_2 = 33°c$

Let us set $x = e^{-h\cdot 10min}$

(and $x^2 = e^{-h\cdot 20min}$).

Then our knowledge is captured as follows:

$$T_1 = T_e + (T_0 - T_e)x$$

$$T_2 = T_e + (T_0 - T_e)x^2$$

The first equation lets us eliminate x:

$$x = \frac{T_1 - T_e}{T_0 - T_e}$$

(we don't need to know it but $x = \frac{39-(-3)}{46-(-3)} = \frac{42}{49} = \frac{6}{7}$)

Put that into the second equation:

$$T_2 - T_e = (T_0 - T_e)\left(\frac{T_1 - T_e}{T_0 - T_e}\right)^2$$

$$(T_2 - T_e)(T_0 - T_e) = (T_1 - T_e)^2$$

$$T_2T_0 - T_0T_e - T_2T_e + \cancel{T_e^2} = T_1^2 - 2T_1T_e + \cancel{T_e^2}$$

(in case you are curious plug back in to find x)

$$T_e = \frac{T_2T_0 - T_1^2}{T_0 + T_2 - 2T_1} = \frac{33\cdot 46 - 39^2}{33+46-2\cdot 39}\,°c = -3°c$$

Also Choose one of the even problems from pp. 34-36.

I'll do #8 (Batch reactor)

concentration

$$\frac{dC}{dt} = -kC$$

$c(t) = C_0 e^{-kt}$ is the general solution. Logan asks when is 90% gone? Or to put that slightly differently, when is $c(t) = 0.1\, C_0$

$$0.1\cancel{C_0} = \cancel{C_0} e^{-kt}$$

$$10 = e^{kt} \qquad kt = \ln 10$$

$$t = \frac{\ln 10}{k} \approx \frac{2.30}{k}$$