ODE Assignment 3

To present Friday, May 13

Problem 1 Logan p. 26 #6

$$\frac{dx}{dt} = (4t-x)^2 \qquad x(0)=1$$

so $y(0)=-1$

Logan suggests introducing $y(t) = 4t - x(t)$

$$\frac{dy}{dt} = 4 - \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = 4 - \frac{dy}{d}$$

$$4 - \frac{dy}{dt} = y^2 \quad \text{or} \quad \int \frac{dy}{4-y^2} = \int dt = t + c$$

$$\int \frac{dy}{4-y^2} = \int \frac{dy}{(2+y)(2-y)} = \int \frac{1}{4}\left(\frac{1}{2+y} + \frac{1}{2-y}\right) dy$$

$$= \frac{1}{4}\left[\ln(2+y) - \ln(2-y)\right] = t + c$$

or $$\ln \frac{2+y}{2-y} = 4t + d$$

or $$\frac{2+y}{2-y} = fe^{4t} \quad \text{or} \quad 2+y = (2-y)fe^{4t}$$

or $$y = \frac{2fe^{4t} - 2}{fe^{4t}+1} = 2\,\frac{fe^{4t}-1}{fe^{4t}+1}$$

$y(0) = -1 \Rightarrow f = \frac{1}{3}$

Problem 1 (CONT'D)

Before going further, let's double-check:

$$\frac{dy}{dt} = 2\,\frac{4fe^{4t}}{fe^{4t}+1} - 2\,\frac{fe^{4t}-1}{(fe^{4t}+1)^2}\,4fe^{4t}$$

$$= 8fe^{4t}\,\frac{(fe^{4t}+1)-(fe^{4t}-1)}{(fe^{4t}+1)^2} = 16fe^{4t}\,\frac{1}{(fe^{4t}+1)^2}$$

whereas

$$4-y^2 = 4\left(1-\frac{f^2e^{8t}-2fe^{4t}+1}{(fe^{4t}+1)^2}\right)$$

$$= 4\,\frac{\cancel{f^2e^{8t}}+2fe^{4t}\cancel{+1}-\cancel{f^2e^{8t}}+2fe^{4t}\cancel{-1}}{(fe^{4t}+1)^2} = 16\,\frac{fe^{4t}}{(fe^{4t}+1)^2}$$

Let's also use that $y(0)=-1 \Rightarrow f=\frac{1}{3}$

So $x(t) = 4t - y(t) = 4t - 2\,\dfrac{fe^{4t}-1}{fe^{4t}+1}$

$$= 4t - 2\,\frac{e^{4t}-3}{e^{4t}+3}$$

The domain of validity is $-\infty < t < \infty$

Problem 2   Logan p. 28 #16

$$\frac{dv}{dt} = g - \frac{av^2}{m}$$

Logan suggests let $r^2 = \frac{ag}{m}$

$$\frac{dv}{dt} = g - \frac{1}{g} r^2 v^2 = g\left(1 - \frac{1}{g^2} r^2 v^2\right)$$

Let $\frac{rv}{g} = w$ $\frac{\cancel{g}}{r}\frac{dw}{dt} = \cancel{g}(1-w^2)$

$$\int \frac{dw}{1-w^2} = \int r\,dt$$

$$\int \frac{1}{2}\left(\frac{1}{1-w} + \frac{1}{1+w}\right) dw = rt + c$$

$$\frac{1}{2}\left(-\ln(1-w) + \ln(1+w)\right) = rt + c$$

$$\ln \frac{1+w}{1-w} = 2(rt+c)$$

$$\frac{1+w}{1-w} = \underbrace{e^{2(rt+c)}}_{\text{just for a moment, call that mess } \alpha}$$

$$1 + w = \alpha(1-w) \qquad w = \frac{\alpha - 1}{\alpha + 1}$$

Problem 2 (CONT'D)

Now Logan suggests letting $v(0) = V$.

Since $w = \frac{rv}{g}$ that means $w(0) = \frac{rV}{g}$

But

$$w = \frac{\alpha - 1}{\alpha + 1} = \frac{e^{2(rt+c)} - 1}{e^{2(rt+c)} + 1}$$

Call e^{2c} $\frac{1}{c_1}$

you'll see why — it is just to agree with Logan

So $$w(0) = \frac{\frac{1}{c_1} - 1}{\frac{1}{c_1} + 1} = \frac{rV}{g}$$

Solve this for $\frac{1}{c_1}$:

$$\frac{1}{c_1} - 1 = \left(\frac{1}{c_1} + 1\right)\frac{rV}{g} \qquad \frac{1}{c_1}\left(1 - \frac{rV}{g}\right) = 1 + \frac{rV}{g}$$

$$\frac{1}{c_1} = \frac{1 + \frac{rV}{g}}{1 - \frac{rV}{g}}$$

Put all this back into the expression for $v(t)$.

$$v(t) = \frac{g}{r} w(t) = \frac{g}{r} \frac{e^{2rt}\frac{1}{c_1} - 1}{e^{2rt}\frac{1}{c_1} + 1}$$

$$= \frac{g}{r} \frac{e^{2rt} - c_1}{e^{2rt} + c_1} = \frac{rm}{a} \frac{e^{rt} - c_1 e^{-rt}}{e^{rt} + c_1 e^{+rt}}$$

The domain of validity is $-\infty < t < \infty$ unless c_1 is negative in which case a problem can develop in the denominator! This can happen if $rV > g$ or $\sqrt{a/m'}\, V > g$.

Problem 3 Logan p. 42 #6

$$x' + \frac{e^{-t}}{t} x = t \qquad x(1) = 0$$

Follow the procedure on p. 38. Multiply both sides by

$$e^{\int_1^t \frac{e^{-s}}{s} ds}$$

$$\underbrace{e^{\int_1^t \frac{e^{-s}}{s} ds} x' + \frac{e^{-t}}{t} e^{\int_1^t \frac{e^{-s}}{s} ds} x}_{\frac{d}{dt}\left(e^{\int_1^t \frac{e^{-s}}{s} ds} x\right)} = e^{\int_1^t \frac{e^{-s}}{s} ds} t$$

$$e^{\int_1^t \frac{e^{-s}}{s} ds} x = \int_1^t e^{\int_1^r \frac{e^{-s}}{s} ds} r \, dr + C_1$$

The initial condition tells us $C_1 = 0$. So

$$x(t) = e^{-\int_1^t \frac{e^{-s}}{s} ds} \int_1^t e^{\int_1^r \frac{e^{-s}}{s} ds} r \, dr$$

I see that there is a mistake on p. 38. Where Logan has $p(t)$ in steps 3 and 4 he should have $q(t)$.

Problem 4 p. 43 #14

$$x'(t) = a(t)x + g(t)x^n$$

Let $\quad y = x^{1-n} \qquad y' = (1-n)x^{-n}x'$

Ok, so multiply the original equation by $(1-n)x^{-n}$

$$\underbrace{(1-n)x^{-n}x'}_{y'} = \underbrace{(1-n)a(t)x^{-n+1}}_{(1-n)a(t)y} + (1-n)g(t)$$

$$y' = (1-n)a(t)y + (1-n)g(t) \qquad x = y^{\frac{1}{1-n}}$$

Exactly what Logan wanted us to show

Problem 5 Logan p. 48 #10

Consider the equation

$$(VE)' = \bar{q}E_{in} - \bar{q}E - kVE^2$$

Let $\quad E = G + \beta$

$$(VG)' = \bar{q}E_{in} - \bar{q}(G+\beta) - kV(G+\beta)^2$$

$$= \underbrace{\bar{q}E_{in} - \bar{q}\beta - kV\beta^2}_{\text{Choose } \beta \text{ such that this term is } 0} \underbrace{- \bar{q}G - 2kVG}_{\text{let } \bar{q}+2kV \text{ be } q} - kVG^2$$

Then

$$(VG)' = -qG - kVG^2$$

So by shifting E and shifting q, we have eliminated the $\bar{q}E_{in}$ term

Problem 5 (CONT'D) Divide through by V

$$C' = -\frac{q}{V}C - kC^2$$

This is a Bernoulli equation with $n=2$
From the previous problem, we are supposed to let a new variable be defined by

$$D = C^{1-n} = C^{1-2} = \frac{1}{C} \qquad D' = -\frac{1}{C^2}C'$$

Multiply the Bernoulli equation by $\frac{-1}{C^2}$

$$-\frac{1}{C^2}C' = \frac{q}{V}\frac{1}{C} + k \quad \text{or} \quad D' = \frac{q}{V}D + k$$

or $D' - \frac{q}{V}D = k$. Multiply by $e^{-qt/V}$ ← the integrating factor

The LHS is then a perfect derivative

$$\frac{d}{dt}\left(e^{-qt/V}D\right) = ke^{-qt/V}$$

So $$e^{-qt/V}D = k\int_0^t e^{-qs/V}ds + \frac{1}{C_0}$$

where in the last step, I have determined the integration constant by using $C(0)=C_0$ and $D(0) = \frac{1}{C(0)} = \frac{1}{C_0}$.

$$e^{-qt/V}D = k\frac{-V}{q}\underbrace{e^{-qs/V}\Big|_0^t}_{e^{-qt/V}-1} + \frac{1}{C_0}$$

Problem 5 (CONT'D)

$$e^{-qt/V} D = k\frac{V}{q}\left(1 - e^{-qt/V}\right) + \frac{1}{C_0}$$

$$D = k\frac{V}{q}\left(e^{qt/V} - 1\right) + \frac{1}{C_0} e^{qt/V}$$

$$= \left(k\frac{V}{q} + \frac{1}{C_0}\right) e^{qt/V} - k\frac{V}{q}$$

$$C(t) = \frac{1}{\left(k\frac{V}{q} + \frac{1}{C_0}\right) e^{\frac{q}{V}t} - k\frac{V}{q}}$$