ODE Assignment 4

To turn in Tuesday, May 17.

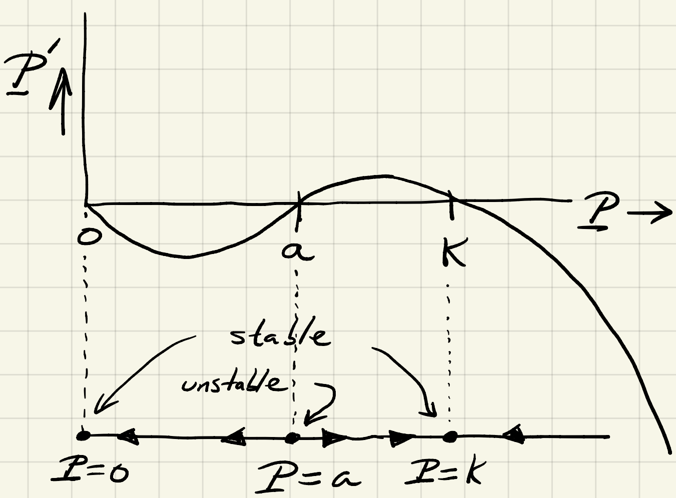
1. p. 63 #4
2. p. 64 #6
3. p. 72 #8
4. p. 72 #10

Problem 1 p. 63 #4 The Allee effect

(a) $P' = rP\left(\frac{P}{a} - 1\right)\left(1 - \frac{P}{K}\right)$ $\quad 0 < a < K$

right-hand side

The RHS is a cubic with three roots. It looks like this:



(b) The roots of the cubic are $P=0$, $P=a$, and $P=K$

(c) 0 and K are stable. $P=a$ is unstable.

(d) A population with insufficient density of mates collapses. In other words, once $P<a$ then $P \to 0$. A population $P>a$ (even very large P, so $a<P<\infty$) either rises toward K if $a<P<K$

(e) The one thing not already stated in parts (a) → (d) is that an extremely large population never leads to a total collapse.

In other words, an extremely large population does not, for example, eat all the food. The system always recovers from excess population with $P \to K$.

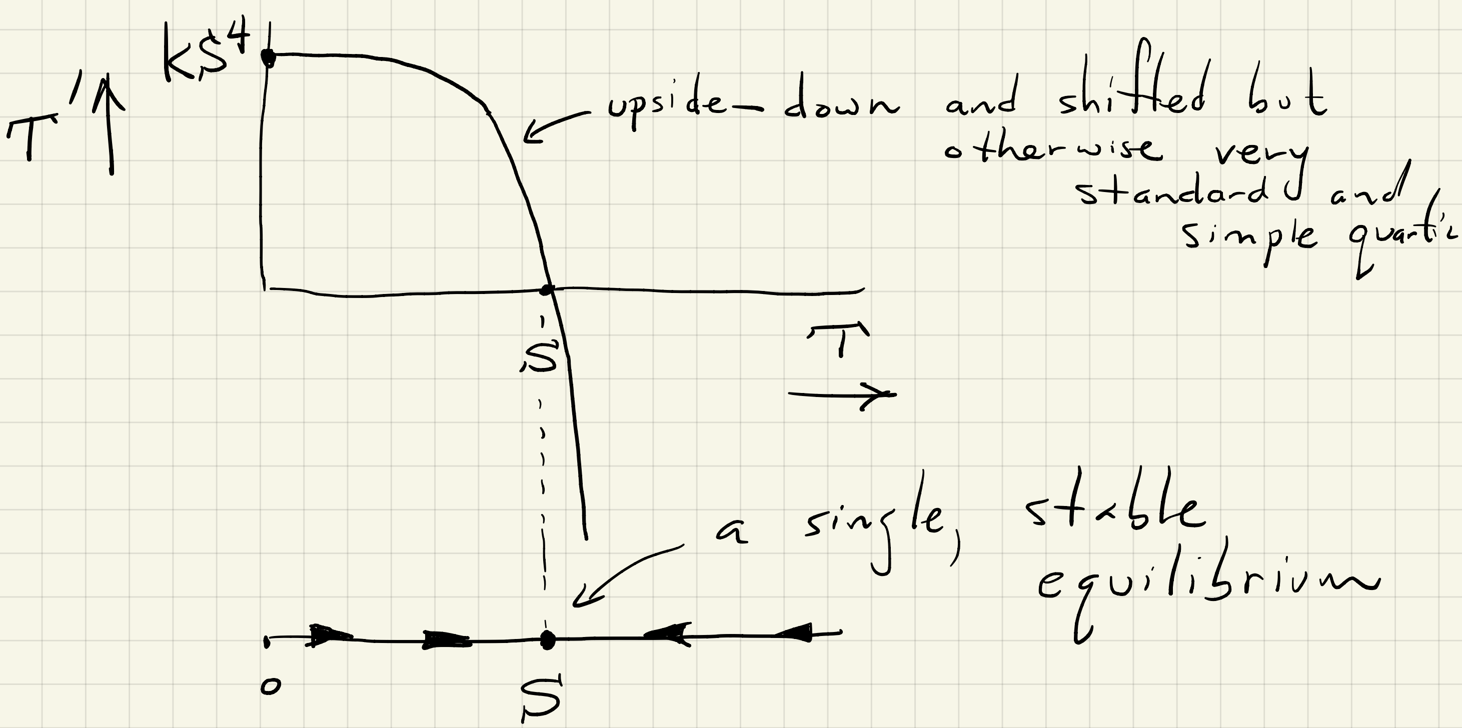
Problem 2. p. 64 #6

$$T' = -k(T^4 - S^4)$$

(a) One root of $T^4 - S^4$ is $T = S$.

So $T^4 - S^4 = (T - S) \cdot$ some cubic

However, it is clear that "some cubic" has two properties: (1) it has no (real) roots, and (2) for $T > 0$ it is always greater than zero.



Problem 2 (CONT'D)

(b) Solve the equation in the limit $T \gg S'$. E.g., neglect S'.

$$T' = -kT^4 \qquad \int \frac{dT}{T^4} = -k\int dt$$

$$-\frac{1}{3}\frac{1}{T^3} = -kt + c \qquad \frac{1}{T^3} = 3kt + d$$

$$T = \frac{1}{\sqrt[3]{3kt+d}} \qquad T(0) = T_0$$

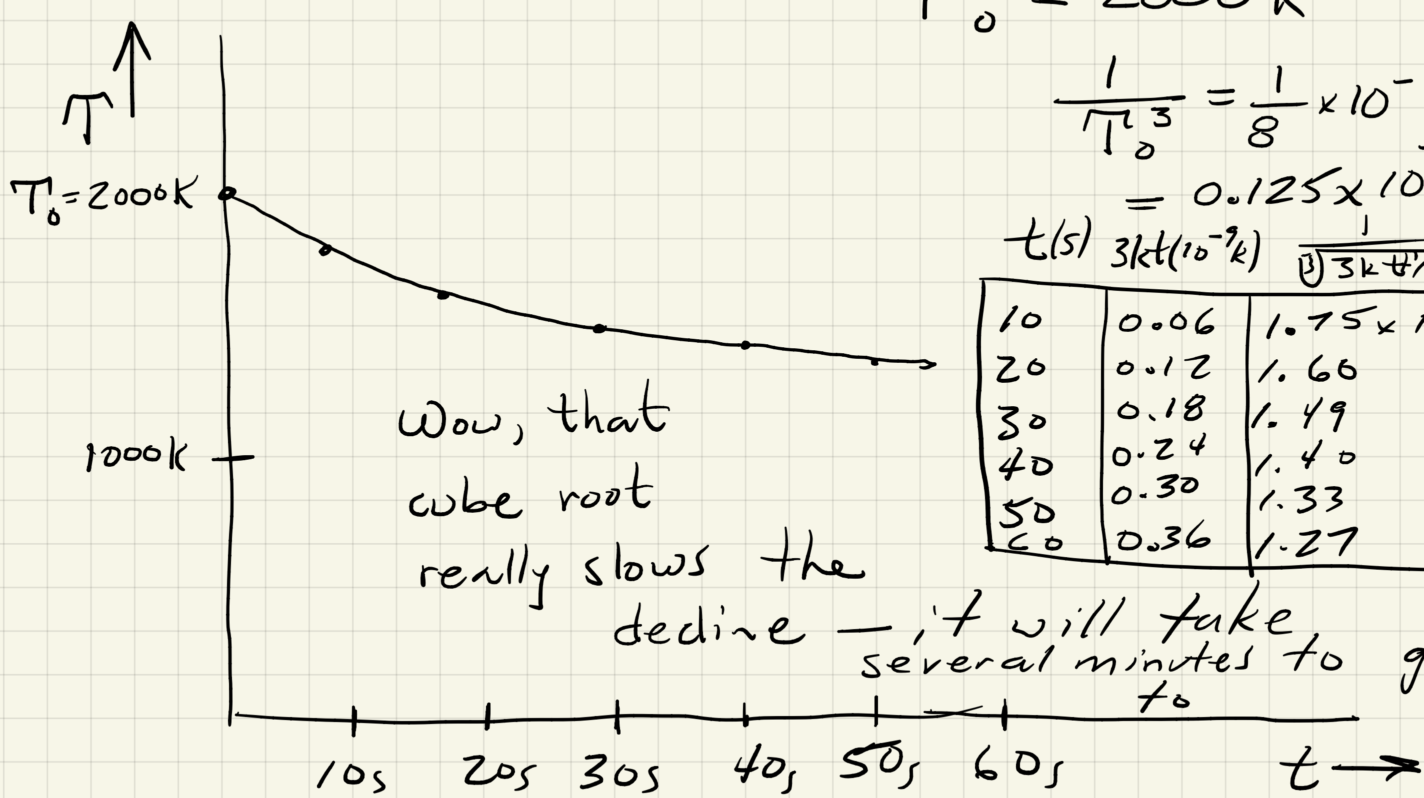
$$T = \frac{1}{\sqrt[3]{3kt + 1/T_0^3}}$$

Plot this with $k = 2\times10^{-12}\,\frac{K^{-3}}{s}$ or $3k = 0.006\times10^{-9}\,\frac{K^{-3}}{s}$

and $T_0 = 2000\,K$

$$\frac{1}{T_0^3} = \frac{1}{8}\times10^{-9}\,K^{-3} = 0.125\times10^{-9}\,K^{-3}$$

t(s)	$3kt$ (10^{-9} K)	$\frac{1}{\sqrt[3]{3kt+1/T_0^3}}$
10	0.06	1.75×10^3 K
20	0.12	1.66
30	0.18	1.49
40	0.24	1.40
50	0.30	1.33
60	0.36	1.27



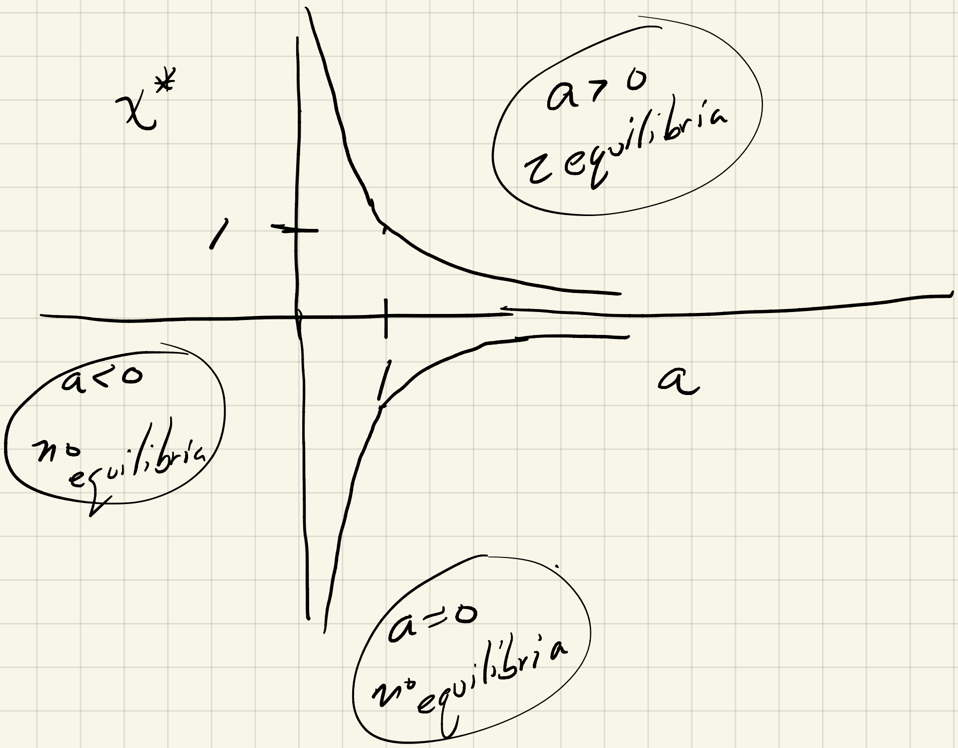
Wow, that cube root really slows the decline — it will take several minutes to get to

Problem 3   p. 72 #8

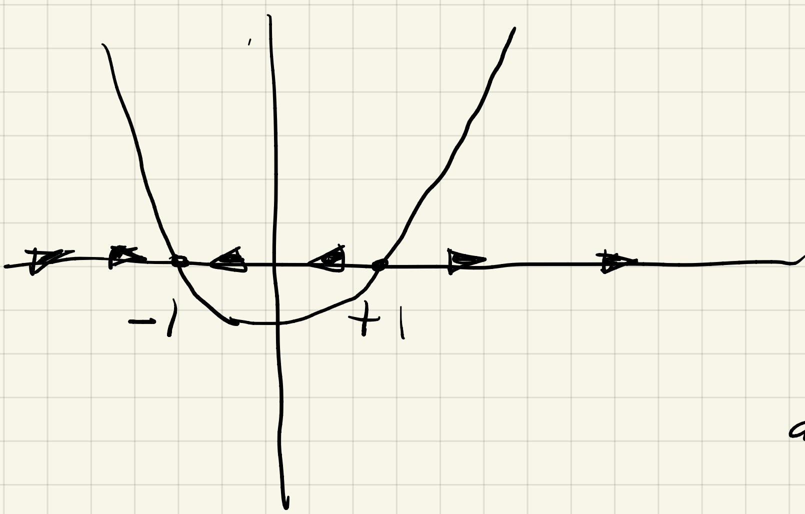
Consider $x' = ax^2 - 1$

Draw the bifurcation diagram

$$ax^2 - 1 = 0 \implies x^* = \pm \frac{1}{\sqrt{a}}$$



Let's take $a=1$ to understand the $a>0$ case better $x' = x^2 - 1$



From this we learn that $x = \frac{1}{\sqrt{a}}$ is unstable and $x = -\frac{1}{\sqrt{a}}$ is stable

Problem 4   p. 72 #10

(a) $$\frac{dP}{d\tau} = I - sP + r\frac{P^n}{M^n + P^n}$$

let $t = s\tau$ $$\frac{dP}{d\tau} = \frac{dP}{dt}\frac{dt}{d\tau}$$

$$p = \frac{P}{M} \qquad = s\frac{dP}{dt}$$

$$sM\frac{dp}{dt} = I - sMp + r\frac{p^n}{1+p^n}$$

or $$\frac{dp}{dt} = a - p + \rho\frac{p^n}{1+p^n}$$

$$a = \frac{I}{sM} \qquad \rho = \frac{r}{sM}$$

(b) Take $n = 8$ and $\rho = 5$

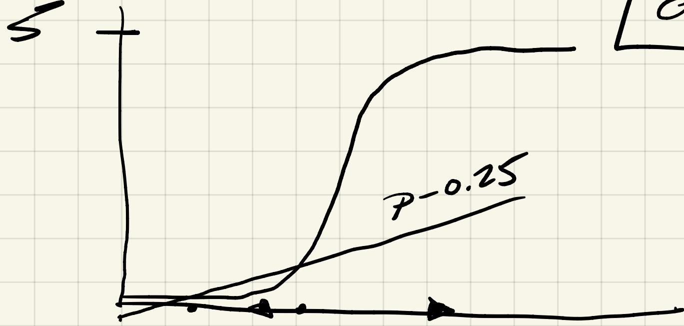
$$\frac{dp}{dt} = a - p + 5\frac{p^8}{1+p^8}$$

I, s, and M are positive, so a is positive. We need to find the roots of the RHS for various values of a.

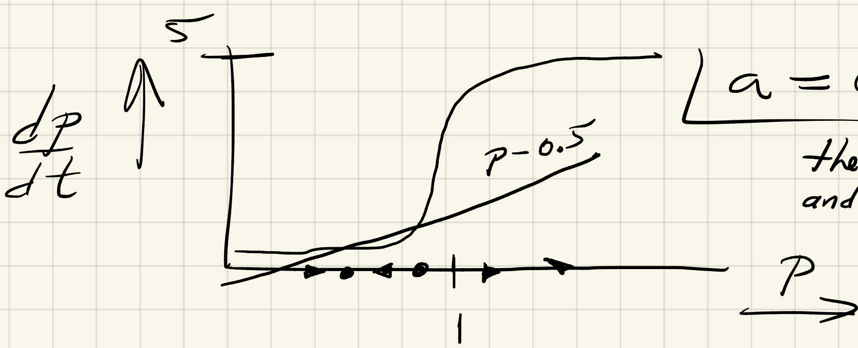
Problem 4 (CONT'D)

(b) $a - p + 5\dfrac{p^8}{1+p^8}$

$a = 0.25$

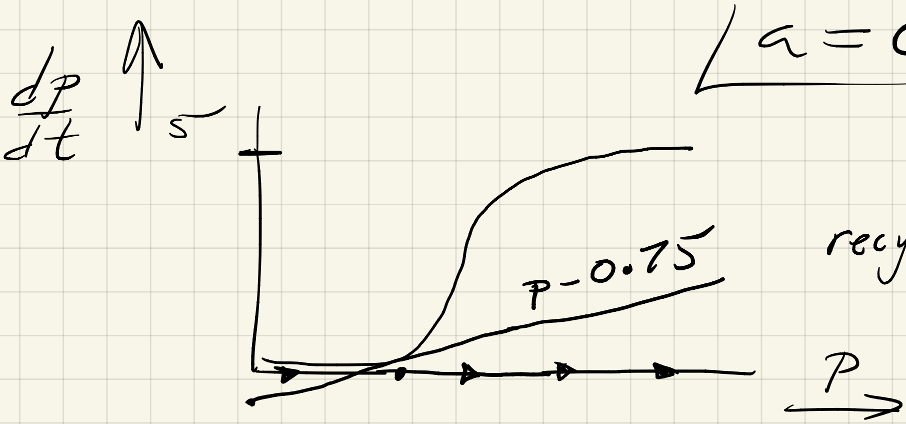


$a = 0.5$



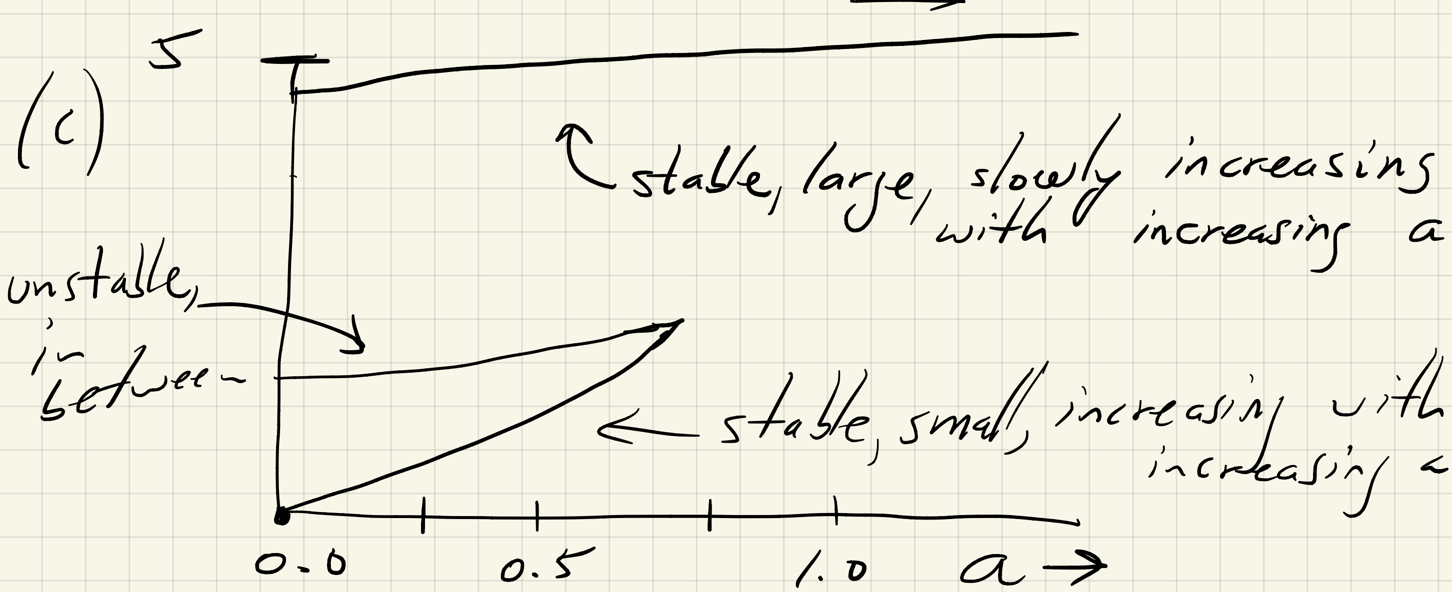
there is a stable p^* and an unstable p^*

$a = 0.75$



recycling always wins

(c)



Explain rapid onset of eutrophication

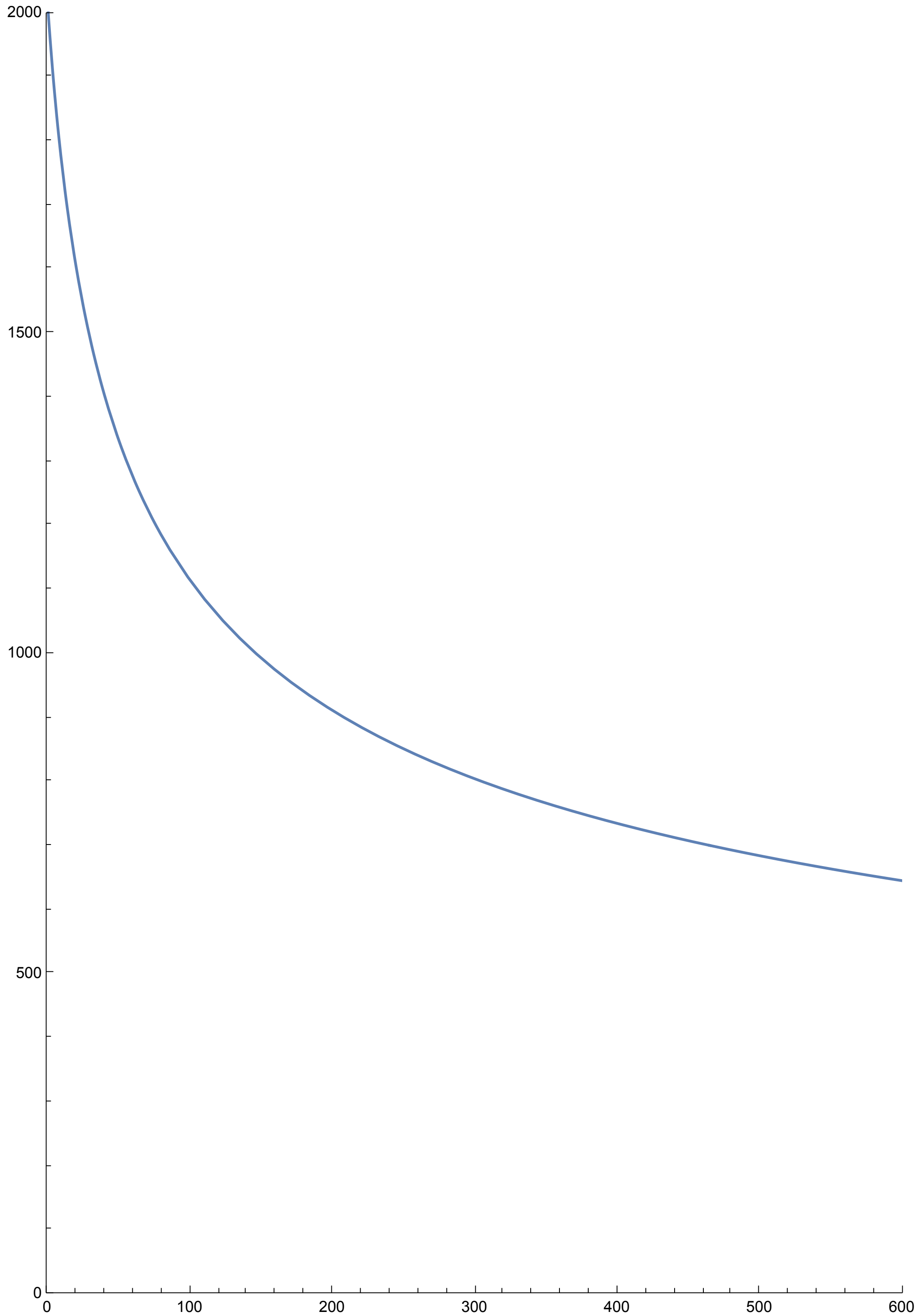
Hmm. There is always either a large p^* or no p^* at all ?

For Problems 2 and 4, let's have Mathematica help us do some nicer graphs than my hand-drawn ones.

In[96]:=
```
temp[t_] := (0.006 t + 0.125)^(-1/3)
```

In[97]:=
```
Plot[1000 temp[t], {t, 0, 600}, PlotRange → {{0, 600}, {0, 2000}}, AspectRatio → 1.5]
```

Out[97]=



In[98]:=
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pMinusA[p_] := p - 0
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In[99]:=
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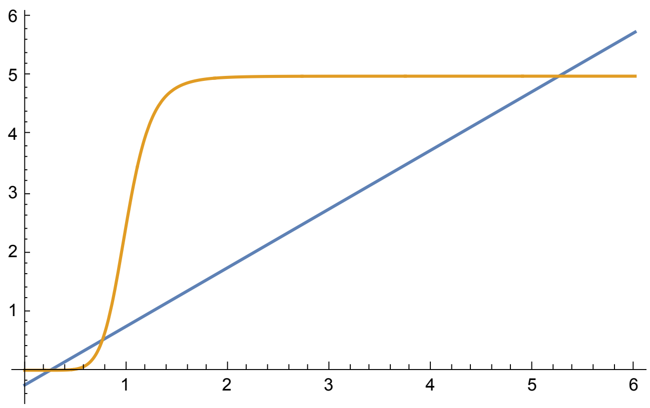
recycling[p_] := 5 p^8 / (1 + p^8)
```

In[50]:=
```
pMinusA[p_, a_] := p - a;
```

In[51]:=
```
recycling[p_] := 5 p^8 / (1 + p^8)
```

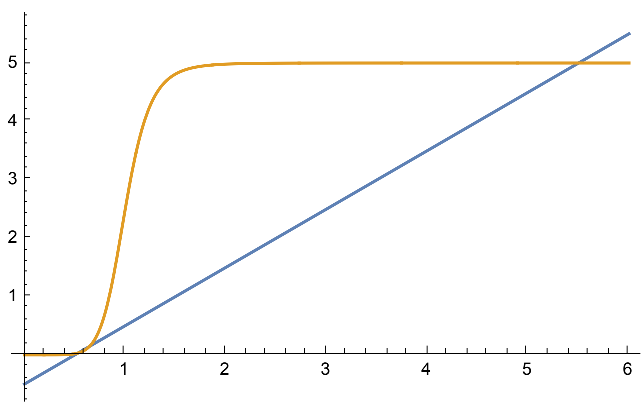
In[52]:=
```
Plot[{pMinusA[p, a] /. a → 0.25, recycling[p]}, {p, 0, 6}]
```

Out[52]=



In[53]:=
```
Plot[{pMinusA[p, a] /. a → 0.5, recycling[p]}, {p, 0, 6}]
```

Out[53]=



In[54]:=
```
Plot[{pMinusA[p, a] /. a → 0.75, recycling[p]}, {p, 0, 6}]
```

Out[54]=