ODE Assignment 5

To present Thursday, May 19

1. p. 77 #4; 2. p. 77 #6; 3. p. 85 #2; 4. p. 85 #3

Problem 1 p. 77 #4

(a) Consider $x' = \underbrace{\frac{-x+e^{-t}}{t(t-5)}}_{f(t,x)}$ with $x(2)=1$

$$\frac{\partial}{\partial x} f(t,x) = \frac{-1}{t(t-5)}$$

Both $f(t,x)$ and $\frac{\partial}{\partial x} f(t,x)$ are not even defined at $t=0$ and $t=5$.

Because this is a first-order linear equation Example 1.41 is applicable. It claims there is a solution for $0<t<5$.

(b) The same reasoning holds for

$$x' + \frac{1}{t-3} x = \frac{1}{t-7} \quad \text{with } x(4)=1$$

There is a solution for $3<t<7$

Problem 2. p.77 #6

(a) Consider $x' = -\frac{4t}{x}$ with $x(0) = x_0$

If $x_0 = 0$, the RHS is $\frac{0}{0}$. We might salvage some meaning in this case by rewriting the equation as

$$x x' = -4t \quad \text{or} \quad \tfrac{1}{2}(x^2)' = -4t$$

or $(x^2)' = -8t$ or $x^2 = -4t^2 + c$

or $x^2 = -4t^2 + x_0^2$

or $$x = \pm\sqrt{x_0^2 - 4t^2}$$

Double-check by differentiating

$$x' = \pm\frac{1}{2}\frac{1}{\sqrt{x_0^2 - 4t^2}}(-8t)$$

$$= \mp\frac{4t}{\sqrt{x_0^2-4t^2}} = \frac{-4t}{x} \quad ✓✓$$

To make sense $x_0^2 - 4t^2 > 0$ because if it is 0 we have 0 in the denominator, and if it is < 0, we can't take the square root.

$x_0^2 > 4t^2$ or $|2t| < |x_0|$

or $-\frac{1}{2}|x_0| < t < \frac{1}{2}|x_0|$

Problem 2 (CONT'D)

(6) Consider $x' + x^3 = 0$ with $x(0) = x_0$

$$\frac{x'}{x^3} = -1 \quad \text{or} \quad -\frac{1}{2}\left(\frac{1}{x^2}\right)' = -1$$

or $\frac{1}{x^2} = 2t + c$ or $\frac{1}{x^2} = 2t + \frac{1}{x_0^2}$

or $x(t) = \pm \frac{1}{\sqrt{2t + \frac{1}{x_0^2}}} \Rightarrow x(t) = \frac{\operatorname{sgn}(x_0)}{\sqrt{2t + 1/x_0^2}}$

NB: ONLY ONE SIGN MATCHES WITH THE INITIAL CONDITION

Double-check by differentiating

$$x' = \left(-\frac{1}{2}\right) \frac{\operatorname{sgn}(x_0)}{\sqrt{2t + \frac{1}{x_0^2}}^{\,3}} \cdot 2$$

$$= \frac{\operatorname{sgn}(x_0)}{\sqrt{2t + \frac{1}{x_0^2}}^{\,3}} = -x^3 \quad \checkmark\checkmark$$

Our solution only makes sense if

$$2t + \frac{1}{x_0^2} > 0 \quad \text{or} \quad t > -\frac{1}{2}\frac{1}{x_0^2}$$

Let us contemplate a special case, namely $x(0) = x_0 = 0$.

Then $x(t) \equiv 0$ works everywhere (meaning $-\infty < t < \infty$).

Problem 2 (CONT'D)

(c) Consider $x' = x^2$ with $x(0) = x_0$

$$\frac{x'}{x^2} = 1 \text{ or } -\left(\frac{1}{x}\right)' = 1$$

or $\left(\frac{1}{x}\right)' = -1$ or $\frac{1}{x} = -t + c$

or $\frac{1}{x} = -t + \frac{1}{x_0}$

or $x = \dfrac{1}{\frac{1}{x_0} - t}$

Double-check by differentiating

$$x' = -\frac{1}{\left(\frac{1}{x_0} - t\right)^2}(-1) = \frac{1}{\left(\frac{1}{x_0} - t\right)^2}$$

$$= x^2 \quad \checkmark\checkmark$$

Our solution is fine except when

$$\frac{1}{x_0} = t$$

So if $x_0 > 0$ $\quad -\infty < t < \frac{1}{x_0}$

and if $x_0 < 0$ $\quad \frac{1}{x_0} < t < \infty$

Let us again contemplate the special case $x(0) = x_0 = 0$. Then $x(t) = 0$ works for $\quad -\infty < t < \infty$ (everywhere).

Problem 3 p. 85 #2

(a) Starting with

$$LQ''+RQ'+\frac{1}{C}Q=0$$

derive the energy dissipation law.

Multiply the entire equation by Q'

$$LQ'Q''+RQ'^2+\frac{1}{C}QQ'=0$$

or $$\frac{1}{2}L\left(Q'^2\right)'+RQ'^2+\frac{1}{2C}\left(Q^2\right)'=0$$

$$\frac{d}{dt}\left(\frac{1}{2}LQ'^2+\frac{1}{2}\frac{1}{C}Q^2\right)=-RQ'^2$$

(b) $E_L=\frac{1}{2}LI^2$ $\quad I=Q'$

= "energy in inductor"

$E_Q=\frac{1}{2}\frac{1}{C}Q^2=$ "energy in the capacitor"

(c) Energy loss is $-RQ'^2=-RI^2$

(d) With no resistor

$$\frac{d}{dt}\left(E_L+E_Q\right)=0$$

Problem 4 p. 85 #3

(a) We make y positive in the downward direction. It is the amount of displacement measured from the equilibrium position of the spring.

By Hooke's "Law"

$$F_{\text{s}} = -ky$$

Gravity pulls downward (the positive direction)

$$F_0 = mg$$

$$ma = \text{mass} \times \text{acceleration} = my''$$

$$my'' = -ky + mg \quad \text{Newton's 2nd Law}$$

(b) Let $x = y - \Delta L$

$$mx'' = -k(x + \Delta L) + mg$$

Choose $-k\Delta L + mg = 0 \Rightarrow \Delta L = \dfrac{mg}{k}$

or $x = y - \dfrac{mg}{k}$

Then

$$mx'' = -kx$$

Problem 4 (CONT'D)

(c) Try $\quad y = x + \Delta L = x + \dfrac{mg}{k}$

in the equation

$$my'' = -ky - \gamma y' + mg$$

$$mx'' = -k\left(x + \frac{\cancel{mg}}{\cancel{k}}\right) - \gamma x' + \cancel{mg}$$

$$= -kx - \gamma x'$$

There is a typo in what Logan wrote. He wrote

$$mx'' = -kx - \gamma x$$

oops