ODE Assignment 6

To turn in Sunday, May 22

1. p. 90 #2; 2. p. 90 #6; 3. p. 94 #2; 4. p. 94 #4

Problem 1 p. 90 #2

(a) $x'' - 4x' + 4x = 0$ with $x(0) = 1$, $x'(0) = 0$

Characteristic equation is

$\lambda^2 - 4\lambda + 4 = 0$ $\quad (\lambda - 2)^2 = 0$

Double root with $\lambda = 2$.

e^{2t} and te^{2t} are both solutions.

Double-check by differentiating. First e^{2t}:

$4e^{2t} - 4 \cdot 2e^{2t} + 4e^{2t} = 0 \leftarrow e^{2t}$ ✓✓

Then te^{2t}:

$(te^{2t})' = e^{2t} + 2te^{2t}$

$(te^{2t})'' = \underbrace{2e^{2t} + 2e^{2t}}_{4e^{2t}} + 4te^{2t}$

$4e^{2t} + 4te^{2t} - 4(e^{2t} + 2te^{2t}) + 4te^{2t} = 0$ ✓✓

So the general solution is $Ae^{2t} + Bte^{2t}$ and the solution that satisfies the initial conditions is $e^{2t} - 2te^{2t}$

Problem 1 (CONT'D)

(b) $x'' - 2x' = 0$ with $x(0)=1$, $x'(0)=0$

$\lambda^2 - 2\lambda = 0$ $\quad \lambda = 0$ or $\lambda = 2$

$x(t) = A + Be^{2t}$

$x'(0) = 0 \Rightarrow B = 0$

$x(0) = 1 \Rightarrow A = 1$

Answer is

$x(t) = 1$

(c) $x'' + 2x' + x = 0$ with $x(0)=1$, $x'(0)=0$

$\lambda^2 + 2\lambda + 1 = 0$ $\quad (\lambda+1)^2 = 0$

Double root with $\lambda = -1$

$x(t) = Ae^{-t} + Bte^{-t}$

$x(0) = 1 \Rightarrow A = 1$

$x'(0) = 0 \Rightarrow B = 1$

Answer is

$x(t) = e^{-t} + te^{-t}$

(d) $x'' + 4x' + 3x = 0$ with $x(0)=1$ $x'(0)=0$

$\lambda^2 + 4\lambda + 3 = 0$ $\quad (\lambda+3)(\lambda+1) = 0$

$\lambda = -3$ or $\lambda = -1$

$x(t) = Ae^{-3t} + Be^{-t}$

$x(0) = 1 \Rightarrow A + B = 1$

$x'(0) = 0 \Rightarrow -3A - B = 0$ or $B = -3A$

$A = -\frac{1}{2}$ $\quad B = \frac{3}{2}$ $\quad x(t) = -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}$

Problem 2 $\quad$ p. 90 #6

$\lambda = 4$ and $\lambda = -6$

are the characteristic values.

So the characteristic equation is

$$(\lambda - 4)(\lambda + 6) = 0 \quad \text{or} \quad \lambda^2 + 2\lambda - 24 = 0$$

So the differential equation is

$$x'' + 2x - 24x = 0$$

Double-check:

$$16 + 8 - 24 = 0 \ \checkmark\checkmark$$

$$36 - 12 - 24 = 0 \ \checkmark\checkmark$$

Problem 3 $\quad$ p. 94 #2

(a) $x'' + x' + 4x = 0$ with $x(0) = 1$, $x'(0) = 0$

$$\lambda^2 + \lambda + 4 = 0 \qquad \lambda = \frac{-1 \pm \sqrt{1 - 16}}{2}$$

$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{15}\, i$$

$$x(t) = e^{-\frac{1}{2}t}\left(A \cos \frac{\sqrt{15}}{2}t + B \sin \frac{\sqrt{15}}{2}t\right)$$

$x(0) = 1 \Rightarrow A = 1.$ $\quad x'(0) = 0 \Rightarrow -\frac{1}{2}A + \frac{\sqrt{15}}{2}B = 0$

or $B = \frac{1}{\sqrt{15}}$.

$$x(t) = e^{-\frac{1}{2}t}\left(\cos \frac{\sqrt{15}}{2}t + \frac{1}{\sqrt{15}} \sin \frac{\sqrt{15}}{2}t\right)$$

Problem 3 (CONT'D)

(b) $x'' - 4x' + 6x = 0$ with $x(0) = 1$, $x'(0) = 0$

$\lambda^2 - 4\lambda + 6 = 0 \qquad \lambda = \dfrac{4 \pm \sqrt{16 - 24}}{2}$

$= 2 \pm \sqrt{2}\, i$

$x(t) = e^{2t}\left(A\cos\sqrt{2}t + B\sin\sqrt{2}t\right)$

$x(0) = 1 \Rightarrow A = 1 \qquad x'(0) = 0 \Rightarrow 2A + \sqrt{2}B = 0$

$B = -\sqrt{2}$

$x(t) = e^{2t}\left(\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t\right)$

(c) $x'' + 9x = 0$ with $x(0) = 1$, $x'(0) = 0$

$\lambda^2 + 9 = 0 \qquad \lambda = \pm 3i$

$x(t) = A\cos 3t + B\sin 3t$

$x(0) = 1 \Rightarrow A = 1$. $x'(0) = 0 \Rightarrow 3B = 0$ or $B = 0$.

$x(t) = \cos 3t$

(d) $x'' - 12x = 0 \qquad \lambda^2 - 12 = 0$

$\lambda = \pm\sqrt{12} \leftarrow$ or $\pm 2\sqrt{3}$, if you prefer

$x(t) = A\cosh\sqrt{12}t + B\sinh\sqrt{12}t$

$x(0) = 1 \Rightarrow A = 1 \qquad x'(0) = 0 \Rightarrow B = 0$

$x(t) = \cosh\sqrt{12}t$

Problem 3 (CONT'D)

(e) $2x'' + 3x' + 3x = 0$ with $x(0) = 1$, $x'(0) = 0$

$$2\lambda^2 + 3\lambda + 3 = 0 \qquad \lambda = \frac{-3 \pm \sqrt{9 - 24}}{4} = -\frac{3}{4} \pm \frac{\sqrt{15}}{4} i$$

$$x(t) = e^{-\frac{3}{4}t}\left(A \cos \frac{\sqrt{15}}{4} t + B \sin \frac{\sqrt{15}}{4} t\right)$$

$$x(0) = 1 \Rightarrow A = 1. \quad x'(0) = 0 \Rightarrow -\frac{3}{4} A + \frac{\sqrt{15}}{4} B = 0$$

$$B = \frac{3}{\sqrt{15}} = \frac{\sqrt{15}}{5}$$

$$x(t) = e^{-\frac{3}{4}t}\left(\cos \frac{\sqrt{15}}{4} t + \frac{\sqrt{15}}{5} \sin \frac{\sqrt{15}}{4} t\right)$$

(f) $\frac{1}{2} x'' + \frac{5}{6} x' + \frac{2}{9} x = 0$ with $x(0) = 1$, $x'(0) = 0$

$$\lambda = -\frac{5}{6} \pm \sqrt{\frac{25}{36} - \frac{4}{9}} = -\frac{5}{6} \pm \underbrace{\sqrt{\underbrace{\frac{25}{36} - \frac{16}{36}}_{9/36}}}_{3/6}$$

$$= -\frac{1}{3} \text{ or } -\frac{4}{3}$$

Double-check

$$\lambda = -\frac{1}{3} \qquad \frac{1}{2}\,\frac{1}{9} + \frac{5}{6}\left(-\frac{1}{3}\right) + \frac{2}{9} = \frac{1}{18} - \frac{5}{18} + \frac{4}{18} = 0 \ \checkmark\checkmark$$

$$\lambda = -\frac{4}{3} \qquad \frac{1}{2}\,\frac{16}{9} + \frac{5}{6}\left(-\frac{4}{3}\right) + \frac{2}{9} = \frac{16}{18} - \frac{20}{18} + \frac{4}{18} = 0 \ \checkmark\checkmark$$

$$x(t) = A e^{-t/3} + B e^{-4t/3}$$

$$x(0) = 1 \Rightarrow A + B = 1. \qquad x'(0) = 0 \Rightarrow -\frac{A}{3} - \frac{4B}{3} = 0$$

or $A = -4B$

$$B = -\frac{1}{3} \qquad A = \frac{4}{3}$$

$$x(t) = \frac{4}{3} e^{-t/3} - \frac{1}{3} e^{-4t/3}$$

Problem 4 p. 94 #4

$\lambda = 5$ and $\lambda = -5$ is the roots so $\lambda^2 - 25 = 0$ is the characteristic equation.

$$x'' - 25x = 0$$

has solution

$$x(t) = A\cosh 5t + B\sinh 5t$$

$$x(0) = 2 \Rightarrow A = 2$$

$$x'(0) = 0 \Rightarrow B = 0$$

$$x(t) = 2\cosh 5t$$