ODE Assignment 7

To present Tuesday, May 24

1. p.99 #2; 2. p.99 #3; 3. p.111 #4; 4. p.111 #6; 5. p.115 #6

Problem 1 p. 99 #2

(a) Find and sketch the solution of

$$x'' + \frac{1}{8}x' + x = 0 \quad \text{with } x(0)=2,\ x'(0)=0$$

Characteristic equation is

$$\lambda^2 + \frac{1}{8}\lambda + 1 = 0 \qquad \lambda = \frac{-\frac{1}{8} \pm \sqrt{1/64 - 4}}{2}$$

$$= -\frac{1}{16} \pm \frac{1}{16}\underbrace{\sqrt{255}}_{\text{a number very slightly less than } 1} i$$

$$= -\frac{1}{16} \pm i\omega$$

$$\omega = 0.998$$

Solution is

$$x(t) = e^{-\frac{1}{16}t}\left(c_1\cos\omega t + c_2\sin\omega t\right)$$

$$x'(t) = -\frac{1}{16}e^{-\frac{1}{16}t}\left(c_1\cos\omega t + c_2\sin\omega t\right) + e^{-\frac{1}{16}t}\left(-c_1\omega\sin\omega t + c_2\omega\cos\omega t\right)$$

$$x'(0)=0 \Rightarrow -\frac{1}{16}c_1 + c_2\omega = 0$$

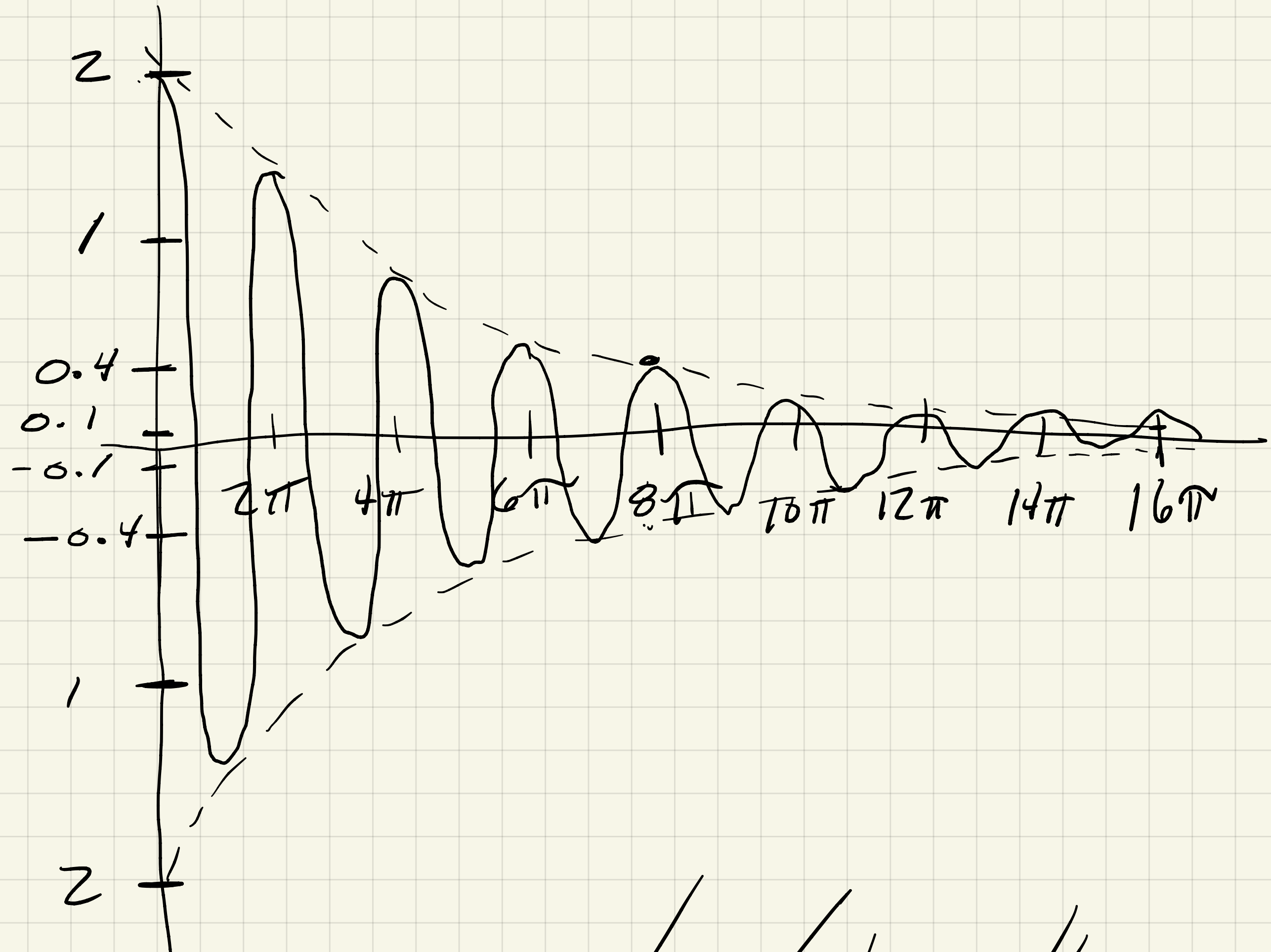
$$x(0)=2 \Rightarrow c_1 = 2 \qquad c_2 = \frac{1}{8}\frac{1}{\omega}$$

Since ω is so close to 1, and even when going out to $t=50$, ωt is still within 0.1 of 50, it seems reasonable to do the sketch with $\omega = 1$.

Problem 1 (CONT'D)

Taking $\omega \approx 1$, our solution is

$$x(t) = e^{-\frac{1}{16}t}\left(2\cos t + \frac{1}{8}\sin t\right)$$



(b) I am going to stick with my $\omega \approx 1$ approximation

$$2^2 + \left(\frac{1}{8}\right)^2 = 4 + \frac{1}{64} = 4\left(1 + \frac{1}{256}\right) = 4\,\frac{257}{256}$$

$$A = 2\sqrt{\frac{257}{256}} \approx 2 \qquad \varphi = \tan^{-1}\frac{1}{16} = 3.6^\circ$$

$$x(t) = 2e^{-t/16}\cos\left(\frac{180}{\pi}t - 4^\circ\right)$$

Problem 2 p. 99 #3

Put $m=1$ $\gamma=1$ in $mx''+\gamma x'+kx=0$

so you just have $x''+x'+kx=0$.

Classify the behavior as the parameter k is varied. To do this, we put in $e^{\lambda t}$ and get the characteristic equation

$$\lambda^2+\lambda+k=0$$

$$\lambda=\frac{-1\pm\sqrt{1-4k}}{2}$$

Decay without oscillation if $1-4k>0$

i.e. if $4k<1$ or $k<\frac{1}{4}$.

Decay with oscillation if $k>\frac{1}{4}$.

Critically damped if $k=\frac{1}{4}$.

No decay only if $\lambda=0$.

Amusingly, you can get that case with $k=0$. The no-decay case has $k=0$ and $\lambda=0$ and is the constant solution $x(t)=x_0$.

Problem 3 p. 111 #4

Solve $x'' - 3x' - 40x = 2e^{-t}$ with $x(0)=0, x'(0)=1$

First solve homogeneous equation.

$$\lambda^2 - 3\lambda - 40 = 0 \qquad \lambda = \frac{3 \pm \sqrt{9 + 4\cdot 1\cdot 40}}{2}$$

$$= \frac{3}{2} \pm \frac{1}{2}\sqrt{169}$$

$$= \frac{3}{2} \pm \frac{13}{2} = 8 \text{ or } -5$$

Now get a solution to the inhomogeneous equation by the method of undetermined coefficients.

$$Ae^{-t}$$

$$A + 3A - 40A = 2 \text{ or } -36A = 2 \text{ or } A = -\frac{1}{18}.$$

So the general solution is

$$c_1 e^{8t} + c_2 e^{-5t} - \frac{1}{18}e^{-t}$$

$$x(0) = 0 \Rightarrow c_1 + c_2 - \frac{1}{18} = 0$$

$$x'(0) = 1 \Rightarrow 8c_1 - 5c_2 + \frac{1}{18} = 1$$

Take $5\times$ the first equation and add it to the second.

$$13c_1 \underbrace{- \frac{5}{18} + \frac{1}{18}}_{-\frac{2}{9}} = 1 \qquad c_1 = \frac{1 + \frac{2}{9}}{13} = \frac{11}{117}$$

$$c_2 = -c_1 + \frac{1}{18} = -\frac{11}{117} + \frac{13}{234} = \frac{-9}{234}$$

$$x(t) = \frac{1}{234}\left(22e^{8t} - 9e^{-5t} - 13e^{-t}\right)$$

Problem 4 p. 111 #6

Solve and plot for $0 \le t \le 30$

$x'' + 2x = \cos\sqrt{2}\,t$ with $x(0)=0$ and $x'(0)=1$

Carrying 2's and $\sqrt{2}$'s around is awkword.

Let $\omega = \sqrt{2}$

$x'' + \omega^2 x = \cos\omega t$ ⟵ Nicer to read

Homogenous solution is

$x = c_1\cos\omega t + c_2\sin\omega t$

Inhomogeneous solution guess

$At\cos\omega t + Bt\sin\omega t$

$$(At\cos\omega t + Bt\sin\omega t)' = A\cos\omega t + B\sin\omega t - A\omega t\sin\omega t + B\omega t\cos\omega t$$

$$(At\cos\omega t + Bt\sin\omega t)'' = -A\omega\sin\omega t + B\omega\cos\omega t - A\omega\sin\omega t + B\omega\cos\omega t - A\omega^2 t\cos\omega t - B\omega^2 t\sin\omega t$$

So

$$(At\cos\omega t + Bt\sin\omega t)'' + \omega^2(At\cos\omega t + Bt\sin\omega t) = -2A\omega\sin\omega t + 2B\omega\cos\omega t = \cos\omega t \Rightarrow \begin{matrix} A=0 \\ B=\frac{1}{2\omega} \end{matrix}$$

Problem 4 (CONT'D)

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t + \frac{1}{2\omega} t \sin\omega t$$

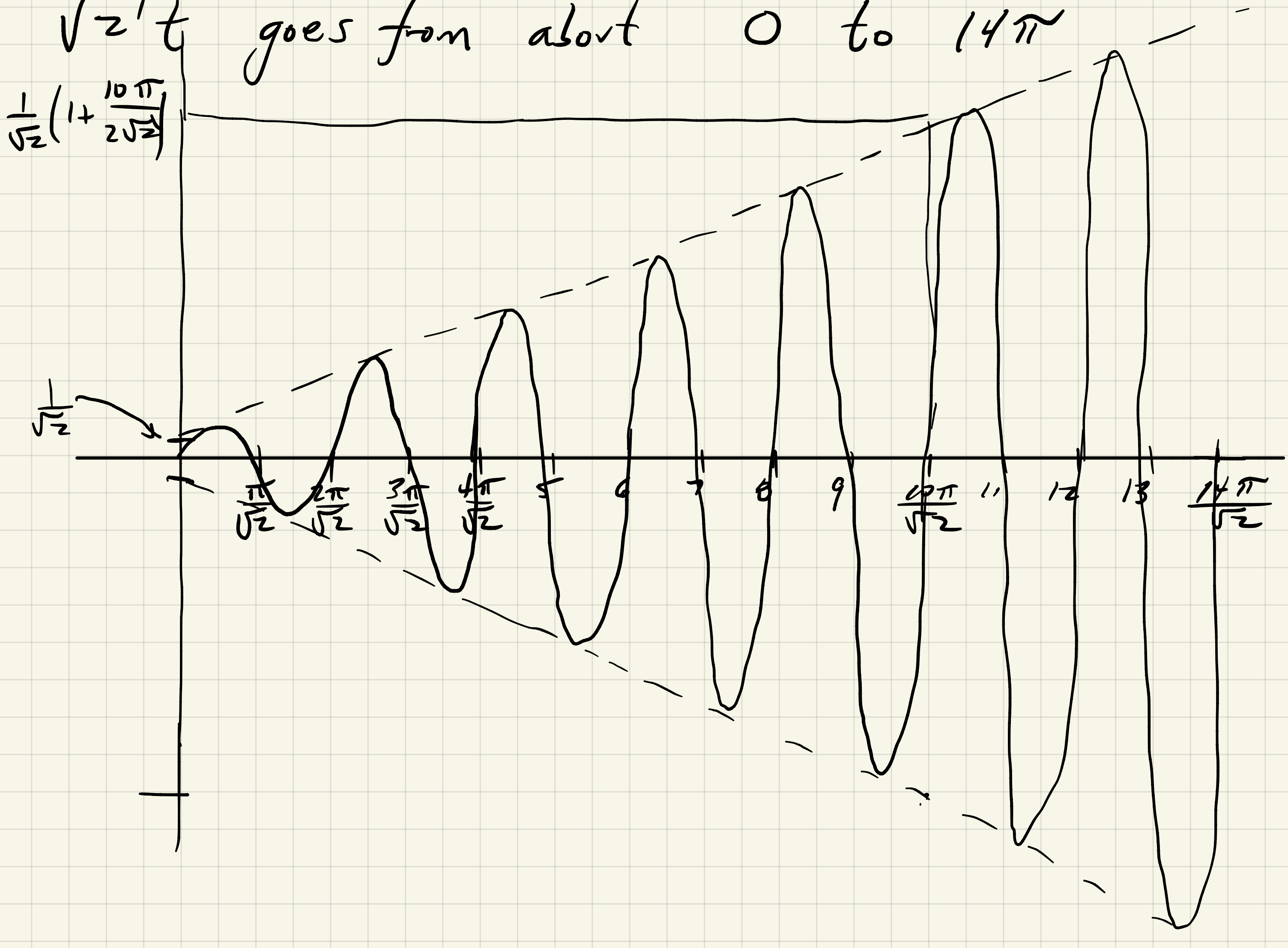
$x(0) = 0 \Rightarrow c_1 = 0$

$x'(0) = 1 \Rightarrow c_2\omega = 1$ or $c_2 = \frac{1}{\omega}$

$$x(t) = \frac{1}{\omega}\sin\omega t + \frac{1}{2\omega} t \sin\omega t$$

Put in $\omega = \sqrt{2}$ and plot for $0 \le t \le 30$

$\sqrt{2}\,t$ goes from about 0 to 14π



Problem 5 p.115 #6

$$x'' + 0.01x' + 4x = \cos 2t$$

Where the 2 and the 4 appear, I will put ω_{00} and ω_{00}^2. To emphasize that $\gamma = 0.01$ is small, I will call it 2ϵ. So we have

$$x'' + 2\epsilon x' + \omega_{00}^2 x = \cos \omega_{00} t \qquad \text{(WOOT!)}$$

Get particular solution of the form

$$A\cos\omega_{00}t + B\sin\omega_{00}t$$

For this solution $x'' + \omega_{00}^2 x = 0$.

So we have

$$2\epsilon\left(-A\omega_{00}\sin\omega_{00}t + B\omega_{00}\cos\omega_{00}t\right) = \cos\omega_{00}t$$

$\Rightarrow A = 0$ and $2\epsilon B\omega_{00} = 1$ or $B = \dfrac{1}{2\epsilon\omega_{00}}$

Now add general solution of the homogeneous equation. First we need its characteristic equation:

$$\lambda^2 + 2\epsilon\lambda + \omega_{00}^2 = 0$$

$$\lambda = \frac{-\cancel{2}\epsilon \pm \sqrt{\cancel{4}\epsilon^2 - \cancel{4}\omega_{00}^2}}{\cancel{2}}$$

Now you see why I stuck the 2 in front of ϵ

Problem 5 (CONT'D)

Can we make an ϵ small approximation?

$$\lambda = -\epsilon \pm i\sqrt{\omega_{00}^2 - \epsilon^2}$$

real part — call this ω_0

General solution of homogeneous equation is

$$x_h(t) = e^{-\epsilon t}\left(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t\right)$$

Meanwhile, we had

$$x_p(t) = \frac{1}{2\epsilon\omega_{00}} \sin \omega_{00} t$$

If $x(0) = 0$, then $c_1 = 0$.

If $x'(0) = 0$, then

$$c_2 \omega_0 + \frac{1}{2\epsilon} = 0 \qquad \text{or} \qquad c_2 = \frac{-1}{2\epsilon\omega_0}$$

So

$$x(t) = x_h(t) + x_p(t)$$

$$= -e^{-\epsilon t}\frac{1}{2\epsilon\omega_0}\sin \omega_0 t + \frac{1}{2\epsilon\omega_{00}}\sin \omega_{00} t$$

This value is annoyingly small for graphing. Did $\epsilon = 0.05$, $2\epsilon = 0.1$ instead.

Now let us put back in

$2\epsilon = 0.01$ and $\omega_{00} = 2$ and graph it.

Plot Solution to Logan p. 115 #6 with ...

... the damping jacked up by a factor of 10 so that the approach to steady state occurs 10x faster.

In[38]:=
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xh[t_] := Exp[-ϵ t] Sin[ω0 t] / (2 ϵ ω0);
xp[t_] := -Sin[ω00 t] / (2 ϵ ω00);
x[t_] := xh[t] + xp[t]
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In[42]:=
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Plot[(xp[t] + xh[t]) /. ω0 → Sqrt[ω00^2 - ϵ^2] /. {ω00 → 2, ϵ → 0.05}, {t, 0, 30 Pi}]
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Out[42]=