ODE Assignment 8

To turn in Thursday, May 26

1. p.115 #7; 2. p.120 #2; 3. p.120 #3; 4. p.124 #6

Problem 1 p.115 #7

Consider $x'' + \omega_0^2 x = \cos \omega t$ ← I have chosen not to use ω and β (instead ω_0, instead ω)

(a) Find the solution when $\beta \neq \omega_0$ with $x(0)=0$ and $x'(0)=0$.

The particular solution of the inhomogeneous equation is $A\cos\omega t$ ($B\sin\omega t$ would only be needed if there was damping).

$$-A\omega^2\cos\omega t + \omega_0^2 A\cos\omega t = \cos\omega t$$

$$A(\omega_0^2-\omega^2)=1 \quad \text{or} \quad A=\frac{1}{\omega_0^2-\omega^2}$$

The general solution of the homogeneous equation is

$$c_1\cos\omega_0 t + c_2\sin\omega_0 t$$

Clearly $c_1=-A$ to satisfy $x(0)=0$

and $c_2=0$ to satisfy $x'(0)=0$

So our solution is

$$x(t)=\frac{1}{\omega_0^2-\omega^2}(\cos\omega t-\cos\omega_0 t)$$

(If you prefer Logan's constants, it is

$$x(t)=\frac{1}{\omega^2-\beta^2}(\cos\beta t-\cos\omega t)\ )$$

Problem 1 (CONT'D)

(b) Rewrite $x(t) = \frac{1}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t)$

Using

$$-2\sin\frac{\omega+\omega_0}{2}t\,\sin\frac{\omega-\omega_0}{2}t$$

$$= \cos\omega t - \cos\omega_0 t$$

This follows from

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$

Let $\alpha = \frac{\omega t + \omega_0 t}{2}$ $\quad \beta = \frac{\omega t - \omega_0 t}{2}$

$$\cos\omega t - \cos\omega_0 t = -2\sin\frac{\omega t + \omega_0 t}{2}\sin\frac{\omega t - \omega_0 t}{2}$$

Double-check with a special case of $\omega = 0$

$$+2\sin^2\frac{\omega_0 t}{2} = 1 - \cos\omega_0 t \leftarrow$$ yes, this is a double-angle formula: $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

So our rewritten answer is

$$x(t) = -2\frac{1}{\omega_0^2 - \omega^2}\sin\frac{\omega+\omega_0}{2}t\,\sin\frac{\omega-\omega_0}{2}t$$

Problem 1 (CONT'D)

(c) Put $\omega_0 = 55$ and $\omega = 45$

Logan calls this ω

It is the natural frequency of the system

Logan calls this β

It is the driving frequency

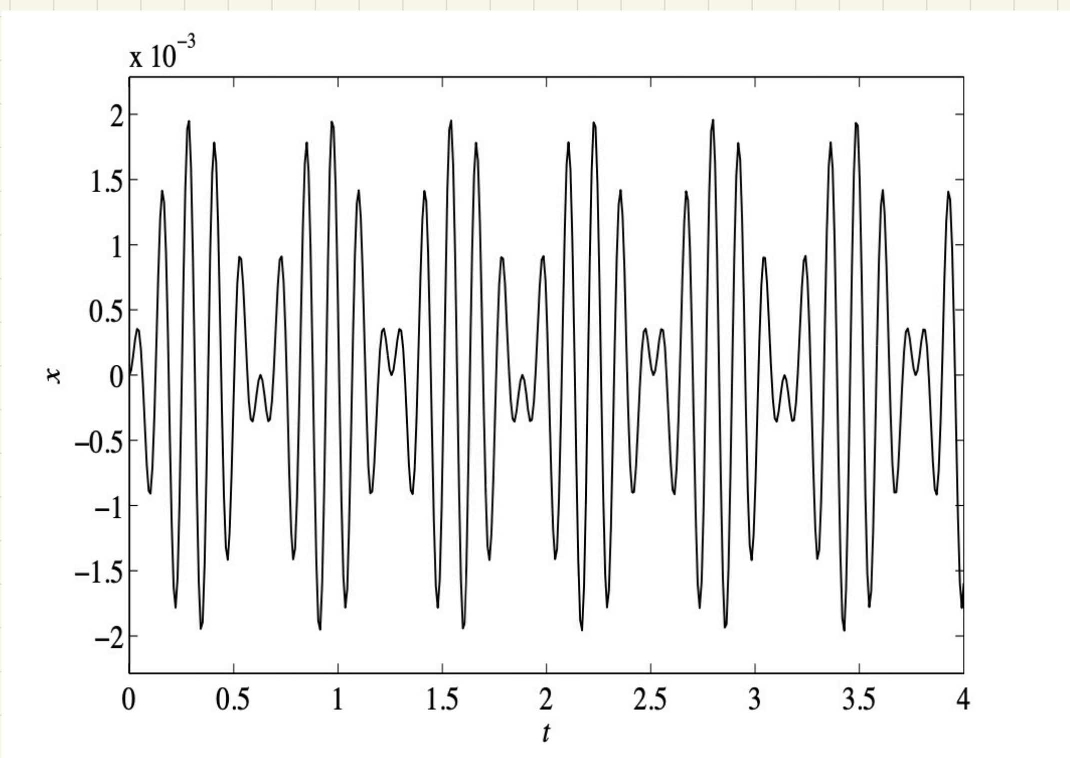
into

$$x(t) = -2\frac{1}{\omega_0^2-\omega^2}\sin\frac{\omega+\omega_0}{2}t\,\sin\frac{\omega-\omega_0}{2}t$$

$$= -2\frac{1}{(\omega_0-\omega)(\omega_0+\omega)}\sin\frac{\omega+\omega_0}{2}t\,\sin\frac{\omega-\omega_0}{2}t$$

$$= +2\frac{1}{10\cdot 100}\sin 50t\,\sin 5t$$

This was plotted by Logan in figure 2.9



Problem 2 p. 120 #2

Solve $x'' + t^2 x' = 0$ with $x(0)=0$ and $x'(0)=1$

This is not a Cauchy-Euler equation, but what if we let $y = x'$? Then it is first-order!

$$y' + t^2 y = 0$$

We solved equations like this in Chapter 1 using integrating factors. Multiply by $e^{\int_0^t s^2 ds}$ and the equation becomes

$$\left(e^{\int_0^t s^2 ds} y\right)' = 0$$

$$e^{\int_0^t s^2 ds} y = c$$

Picking $c=1$ makes $y(0) = x'(0) = 1$

So $y = e^{-\int_0^t s^2 ds}$

And $x = \int_0^t e^{-\int_0^r s^2 ds} dr$

(Choosing the lower limit of the r integration to be 0 gets us $x(0)=0$.)

Despite lots of effort, I can only simplify this as far as

$$\boxed{x(t) = \int_0^t e^{-r^3/3} dr \longleftarrow \text{Boooo!}}$$

Double check $x'(t) = e^{-t^3/3}$

$$x''(t) = -t^2 e^{-t^3/3} = -t^2 x'(t) \checkmark$$

Problem 3   Logan p. 120 #3

Since Logan p.120 #1 wasn't assigned, but we are going to use a new method to solve 1(a), let's first solve it using the methods of Subsection 2.4-1:

$$t^2x'' + x = 0 \qquad x = t^m \qquad x'' = m(m-1)t^{m-2}$$

$$m(m-1)+1=0 \text{ or } \left(m-\tfrac12\right)^2 - \tfrac14 + 1 = 0$$

$$m = \tfrac12 \pm \sqrt{\tfrac34}\, i$$

$$x(t) = t^{1/2}\left[c_1\cos\left(\tfrac{\sqrt3}{2}\ln t\right) + c_2 \sin\left(\tfrac{\sqrt3}{2}\ln t\right)\right]$$

Double-check

$$\left(t^{1/2}\cos\left(\tfrac{\sqrt3}{2}\ln t\right)\right)'' = \tfrac12\left(-\tfrac12\right)t^{-3/2}\cos\left(\tfrac{\sqrt3}{2}\ln t\right)$$
$$\cancel{-2\cdot\tfrac12\, t^{-1/2}\sin\left(\tfrac{\sqrt3}{2}\ln t\right)\cdot\tfrac{\sqrt3}{2}\tfrac1t}$$
$$+\, t^{1/2}\underbrace{\left(-\sin\left(\tfrac{\sqrt3}{2}\ln t\right)\cdot\tfrac{\sqrt3}{2}\tfrac1t\right)'}_{-\cos\left(\frac{\sqrt3}{2}\ln t\right)\cdot\frac{\sqrt3}{2}\frac1t\frac{\sqrt3}{2}\frac1t\ \cancel{+\sin\left(\frac{\sqrt3}{2}\ln t\right)\cdot\frac{\sqrt3}{2}\frac{1}{t^2}}}$$

$$= \left(-\tfrac14 - \tfrac34\right)t^{-3/2}\cos\left(\tfrac{\sqrt3}{2}\ln t\right) = -\frac{1}{t^2}\, t^{1/2}\cos\left(\tfrac{\sqrt3}{2}\ln t\right) \checkmark\checkmark$$

$$\left(t^{1/2}\sin\left(\tfrac{\sqrt3}{2}\ln t\right)\right)'' = \tfrac12\left(-\tfrac12\right)t^{-3/2}\sin\left(\tfrac{\sqrt3}{2}\ln t\right)$$
$$\cancel{+2\cdot\tfrac12\, t^{-1/2}\cos\left(\tfrac{\sqrt3}{2}\ln t\right)\cdot\tfrac{\sqrt3}{2}\tfrac1t}$$
$$+\, t^{1/2}\underbrace{\left(\cos\left(\tfrac{\sqrt3}{2}\ln t\right)\cdot\tfrac{\sqrt3}{2}\tfrac1t\right)'}_{-\sin\left(\frac{\sqrt3}{2}\ln t\right)\cdot\frac{\sqrt3}{2}\frac1t\frac{\sqrt3}{2}\frac1t\ \cancel{-\cos\left(\frac{\sqrt3}{2}\ln t\right)\cdot\frac{\sqrt3}{2}\frac{1}{t^2}}}$$

$$= \left(-\tfrac14 - \tfrac34\right)t^{-3/2}\sin\left(\tfrac{\sqrt3}{2}\ln t\right) = -\frac{1}{t^2}\, t^{1/2}\sin\left(\tfrac{\sqrt3}{2}\ln t\right) \checkmark\checkmark$$

Problem 3 (CONT'D)

Ok, so we know the answer, let's find it the new way $\quad \tau = \ln t \quad$ so $\quad \tau' = \frac{1}{t}$

$$\left.\begin{aligned} x' &= \tau' \frac{dx}{d\tau} = \frac{1}{t}\frac{dx}{d\tau} \\ x'' &= -\frac{1}{t^2}\frac{dx}{d\tau} + \frac{1}{t}\tau'\frac{d^2x}{d\tau^2} \\ &= -\frac{1}{t^2}\frac{dx}{d\tau} + \frac{1}{t^2}\frac{d^2x}{d\tau^2} \end{aligned}\right\} \text{These are the critical steps}$$

So $\quad at^2x'' + btx' + cx$

$$= a\left(\frac{d^2x}{d\tau^2} - \frac{dx}{d\tau}\right) + b\frac{dx}{d\tau} + cx$$

$$= a\frac{d^2x}{d\tau^2} + (b-a)\frac{dx}{d\tau} + cx = 0$$

Solve Logan p. 120 #1(a) as a special case.

we have $\quad a=1 \quad b=0 \quad c=1$

So $\quad t^2x'' + x = \frac{d^2x}{dt^2} - \frac{dx}{d\tau} + x = 0$

Characteristic equation is

$$\lambda^2 - \lambda + 1 = 0 \quad \text{or} \quad \left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4} + 1 = 0$$

Or $\quad \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

So $\quad x(\tau) = e^{\frac{1}{2}\tau}\left(c_1 \cos\frac{\sqrt{3}}{2}\tau + c_2 \sin\frac{\sqrt{3}}{2}\tau\right)$

Or $\quad x(t) = \sqrt{t}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}\ln t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln t\right)\right)$ ✓

Problem 4 Logan p. 124 #6

The result of Logan p. 124 #5 is that if $L\phi = 0$ for some ϕ then $Lx = f$ has as a solution

$$x(t) = \int_a^t \phi(t-s) f(s) ds + x(a)$$

In this problem, we want to apply this result to solve $x'' - 2mx' + m^2 x = f(t)$.

The hint is to try $\phi = te^{mt}$

$$\phi' = e^{mt} + mte^{mt}$$

$$\phi'' = me^{mt} + m^2te^{mt} + me^{mt} = m^2te^{mt} + 2me^{mt}$$

$$\phi'' - 2m\phi' + m^2\phi = \cancel{m^2te^{mt}} + \cancel{2me^{mt}} - 2m(\cancel{e^{mt}} + \cancel{mte^{mt}}) + \cancel{m^2te^{mt}} = 0 \checkmark\checkmark$$

So

$$x(t) = \int_a^t (t-s)e^{m(t-s)} f(s) ds$$

This is a solution satisfying $Lx = f$ with a being any constant and $x(a) = 0$.

We are done with Logan p. 124 #6, but see the next page.

Problem 4 (CONT'D)

As a bonus, I should do p. 124 #5, since we used that result for p. 124 #6.

$$x_p(t) = \int_a^t \phi(t-s)f(s)\,ds + x_p(a)$$

$$x_p'(t) = \underbrace{\phi(t-t)}_{\substack{\text{0 because we} \\ \text{assume } \phi(0)=0}} f(t) + \int_a^t \phi'(t-s)f(s)\,ds$$

$$x_p''(t) = \underbrace{\phi'(t-t)}_{\substack{\text{1 because} \\ \text{we assume} \\ \phi'(0)=1}} f(t) + \int_a^t \phi''(t-s)\,ds$$

Therefore

$$x_p'' + p x_p' + q x_p$$

$$= \int_a^t \Big(\underbrace{\phi(t-s) + p\,\phi'(t-s) + q\,\phi''(t-s)}_{0}\Big) f(s)\,ds + f(t)$$

$$= f(t)$$ ☺