ODE Assignment 9

To present Tuesday, May 31

1. p.145 #6; 2. p.145 #8; 3. p.146 #10

Problem 1 Logan p.145 #6

Show $\mathcal{L}[f(t)H(t-a)] = e^{-as}\mathcal{L}[f(t+a)]$

We start with the definition

$$\mathcal{L}[f(t)H(t-a)](s) \equiv \int_0^\infty f(t)H(t-a)e^{-st}dt$$

$$= \int_a^\infty f(t)e^{-st}dt$$ ← Because the Heaviside function is 0 when $t<a$

$$= \int_0^\infty f(t'+a)e^{-s(t'+a)}dt'$$ ← change of integration variable $t' = t-a$ or $t'+a = t$

$$= e^{-sa}\int_0^\infty f(t+a)e^{-st}$$

drop primes and pull out e^{-sa}

$$= e^{-sa}\mathcal{L}[f(t+a)](s)$$

Therefore

$$\mathcal{L}[f(t)H(t-a)] = -e^{sa}\mathcal{L}[f(t+a)]$$

☺

Problem 2   Logan p. 145 #8

Find the Laplace transform of each of the following eight functions:

(a) $\mathcal{L}\left[6+5e^{-2t}+te^{3t}\right](s)$

$$= \int_0^\infty \left(6+5e^{-2t}+te^{3t}\right)e^{-st}\,dt$$

we have an issue — this term blows up unless $s>3$

$$= 6\frac{1}{s} + 5\frac{1}{s+2} - \frac{d}{ds}\underbrace{\int_0^\infty e^{(3-s)t}\,dt}_{\frac{1}{s-3}}$$

$$= 6\frac{1}{s} + 5\frac{1}{s+2} + \frac{1}{(s-3)^2}$$ provided $s>3$

(b) $\mathcal{L}\left[t\,H(t-3)\right](s)$

$$= \int_0^\infty t\,H(t-3)e^{-st}\,dt$$

$$= \int_3^\infty te^{-st}\,dt = -\frac{d}{ds}\int_3^\infty e^{-st}\,dt$$

$$= -\frac{d}{ds}\int_0^\infty e^{-s(t+3)}\,dt = -\frac{d}{ds}\left(e^{-3s}\frac{1}{s}\right)$$

$$= 3e^{-3s}\frac{1}{s} + \frac{1}{s^2}e^{-3s} = \frac{1}{s^2}(1+3s)e^{-3s}$$

Problem 2 (CONT'D)

(c) $$\mathcal{L}[\cos 5t](s) = \int_0^\infty \frac{e^{5it}+e^{-5it}}{2} e^{-st}\,dt$$

$$= \frac{1}{2}\left(\frac{1}{s-5i} + \frac{1}{s+5i}\right) = \frac{1}{2}\,\frac{2s}{s^2+25}$$

(d) Well $\sin(\alpha+\pi) = -\sin\alpha$, so

$$\mathcal{L}[\sin(2t+\pi)](s) = -\int_0^\infty \sin 2t\, e^{-st}\,dt$$

$$= -\frac{1}{2i}\int_0^\infty \left(e^{2it}-e^{-2it}\right)e^{-st}\,dt$$

$$= -\frac{1}{2i}\left(\frac{1}{s-2i} - \frac{1}{s+2i}\right)$$

$$= -\frac{1}{2i}\,\frac{4i}{s^2+4} = -\frac{2}{s^2+4}$$

(e) $$\mathcal{L}[3e^{-t}\cosh t](s) = \frac{3}{2}\int_0^\infty e^{-t}\left(e^{-t}+e^{t}\right)e^{-st}\,dt$$

$$= \frac{3}{2}\left(\frac{1}{s+2} + \frac{1}{s}\right)$$

(f) Well $\cos(\alpha-\pi) = -\cos\alpha$, so

$$\mathcal{L}[H(t-\pi)\cos(t-\pi)](s)$$

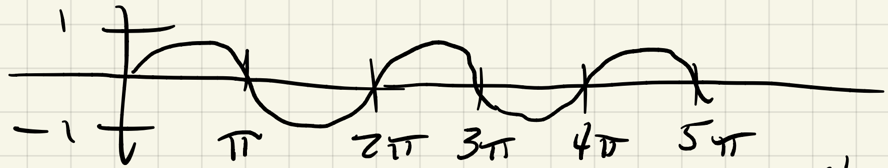
$$= -\int_\pi^\infty \cos t\, e^{-st}\,dt = -\frac{1}{2}\int_\pi^\infty \left(e^{it}+e^{-it}\right)e^{-st}\,dt$$

$$= -\frac{1}{2}\left(\frac{1}{s-i}\,e^{i\pi-s\pi} + \frac{1}{s-i}\,e^{-i\pi-s\pi}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right)e^{-s\pi} = \frac{s}{s^2+1}\,e^{-s\pi}$$

Problem 3 Logan p. 146 #10

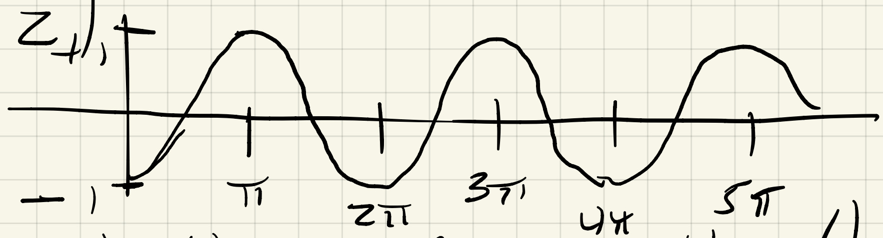
(a) $\sin t$



$$\mathcal{L}[\sin t](s) = \int_0^\infty \frac{1}{2i}\left(e^{it} - e^{-it}\right)e^{-st}\,dt$$

$$= \frac{1}{2i}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) = \frac{1}{s^2+1}$$

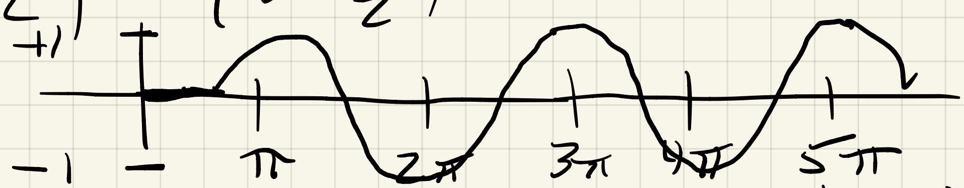
(b) $\sin\left(t - \frac{\pi}{2}\right)$



$$\mathcal{L}\left[\sin\left(t-\frac{\pi}{2}\right)\right](s) = -\frac{1}{2}\int_0^\infty \left(e^{it} + e^{-it}\right)e^{-st}\,dt$$

$$= -\frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) = -\frac{s}{s^2+1}$$

(c) $H\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right)$



$$\mathcal{L}[\quad](s) = -\frac{1}{2}\int_{\frac{\pi}{2}}^\infty \left(e^{it} + e^{-it}\right)e^{-st}\,dt$$

$$= -\frac{1}{2}\left(\frac{e^{i\frac{\pi}{2}}e^{-s\frac{\pi}{2}}}{s-i} + \frac{e^{-i\frac{\pi}{2}}e^{-s\frac{\pi}{2}}}{s+i}\right)$$

$$= -\frac{1}{2}\left(\frac{i}{s-i} - \frac{i}{s+i}\right)e^{-s\frac{\pi}{2}} = \frac{1}{s^2+1}\,e^{-s\frac{\pi}{2}}$$