OPE Assignment 10

To turn in Thursday, June 2nd
Logan p. 157 #8, #9, and #10
and any two of
Logan p. 158 #11, #12, and #13

Problem 1 Logan p. 157 #8

Find the Laplace transform of $f(t) = t^2 H(t-3)$

$$\mathcal{L}[f](s) = \int_0^\infty t^2 H(t-3) e^{-st}\,dt$$

$$= \int_3^\infty t^2 e^{-st}\,dt = e^{-3s}\int_0^\infty (t+3)^2 e^{-st}\,dt$$

$$= e^{-3s}\left(\int_0^\infty t^2 e^{-st}\,dt + 6\int_0^\infty t e^{-st}\,dt + 9\int_0^\infty e^{-st}\,dt\right)$$

$$= e^{-3s}\left(\frac{d^2}{ds^2} - 6\frac{d}{ds} + 9\right)\underbrace{\int_0^\infty e^{-st}\,dt}_{\frac{1}{s}}$$

$$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$$

Problem 2 Logan p. 157 #9

Invert $F(s) = \dfrac{1}{(s-2)^4}$

Well, $\displaystyle\int_0^\infty e^{2t} e^{-st}\,dt = \frac{1}{s-2}$

So if I take $-\dfrac{d^3}{ds^3}$ of both sides, I get

$$\int_0^\infty t^3 e^{2t} e^{-st}\,dt = \overbrace{-(-1)(-2)(-3)}^{6}\frac{1}{(s-2)^4}$$

Therefore, $f(t) = \frac{1}{6} t^3 e^{2t}$

Problem 3 Logan p. 158 #10

Invert $F(s) = \dfrac{1-e^{-4s}}{s^2}$

The $\frac{1}{s^2}$ term is easy. Its inverse is t.

use the usual differentiation trick as was just done in the previous problem

what about $\dfrac{e^{-4s}}{s^2}$?

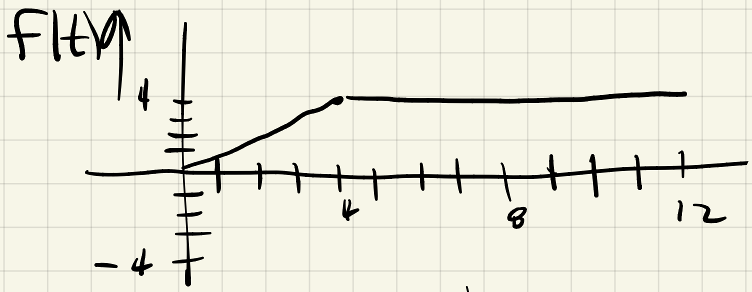
Well $\displaystyle\int_0^\infty H(t-4)e^{-st}\,dt = \int_4^\infty e^{-st}\,dt = e^{-4s}\frac{1}{s}$

Take $-\frac{d}{ds}$ of both sides $\displaystyle\int_0^\infty t H(t-4) e^{-st}\,dt = 4e^{-4s}\frac{1}{s} + \frac{1}{s^2}e^{-4s}$

So

$$\mathcal{L}^{-1}\left[\frac{1-e^{-4s}}{s^2}\right](t) = t - \left[t H(t-4) - 4H(t-4)\right]$$
$$= t - (t-4)H(t-4)$$

Problem 3 (CONT'D) Graph it.



We are done, but I want to double-check. Let's do the integral of the function we have graphed. It is:

$$\int_0^4 t\, e^{-st}\, dt + 4 \int_4^\infty e^{-st}\, dt$$

$$= -\frac{d}{ds} \underbrace{\int_0^4 e^{-st}\, dt}_{\frac{-1}{s}(e^{-4s}-1)} + 4 \underbrace{\int_0^\infty e^{-st}\, dt}_{\frac{1}{s}} - 4 \underbrace{\int_0^4 e^{-st}\, dt}_{\frac{-1}{s}(e^{-4s}-1)}$$

$$= \frac{d}{ds}\left(\frac{1}{s}\left(e^{-4s}-1\right) \right) + 4\frac{1}{s} + 4\,\frac{1}{s}\left(e^{-4s}-1\right)$$

$$= -\frac{1}{s^2}\left(e^{-4s}-1\right) \cancel{- 4\frac{1}{s} e^{-4s}} + \cancel{4\frac{1}{s}} + 4\frac{1}{s}\left(\cancel{e^{-4s}}\cancel{-1}\right)$$

$$= \frac{1}{s^2}\left(1 - e^{-4s}\right)$$

Given all the possible sources of algebraic error, it is nice to have this double-check.

Problem 4   Logan p. 158 #11

Solve $x'' + k^2 x = \left(1 - H\left(t - \frac{4\pi}{k}\right)\right)\cos kt$

with $x(0) = 0$, $x'(0) = 0$, and $k = 2$.

Laplace transform the equation to get

$$s^2 X(s) - s\cancelto{0}{x(0)} - \cancelto{0}{x'(0)} + k^2 X(s)$$

$$= \mathcal{L}\left[\left(1 - H\left(t - \frac{4\pi}{k}\right)\right)\cos kt\right](s)$$

$$X(s) = \frac{1}{s^2+k^2}\mathcal{L}\left[\left(1 - H\left(t - \frac{4\pi}{k}\right)\right)\underbrace{\cos kt}_{\frac{1}{2}\left(e^{ikt}+e^{-ikt}\right)}\right](s)$$

$$= \frac{1}{s^2+k^2}\Bigg[\underbrace{\frac{1}{2}\left(\frac{1}{s-ik} + \frac{1}{s+ik}\right)}_{\substack{\text{first term was easy — because} \\ \text{we have done } \mathcal{L}[\cos] \\ \text{a few times already}}}$$

$$- \frac{1}{s^2+k^2}\mathcal{L}\Bigg[\underbrace{H\left(t - \frac{4\pi}{k}\right)\cos k\left(t - \frac{4\pi}{k} + \cancel{\frac{4\pi}{k}}\right)}_{\substack{\text{rewritten so I can} \\ \text{use the shift pattern}}}\Bigg](s)$$

$$= \frac{1}{s^2+k^2}\left[\frac{s}{s^2+k^2} - e^{-4\pi s/k}\frac{s}{s^2+k^2}\right]$$

So now I have to invert that!?

This is doable, but let's try another approach and then get back to this one.

Problem 4 (CONT'D) $\boxed{\text{Another approach.}}$

From $t=0$ to $t=4\pi/k$

This is a nice ordinary resonance problem.

$$x''+k^2x=\cos kt \qquad \text{with} \qquad \begin{array}{l} x(0)=0 \\ x'(0)=0 \\ k=2 \end{array}$$

The particular solution is

$$At\sin kt$$

$$2Ak\cos kt=\cos kt \qquad A=\frac{1}{2k}$$

$$x_p(t)=\frac{1}{2k}t\sin kt$$

This solution already satisfies

$x_p(t)=0$ and $x_p'(t)=0$

so we don't need to add in any homogeneous solution.

When $t=4\pi/k$

$$x_p\left(\frac{4\pi}{k}\right)=\frac{1}{2k}\frac{4\pi}{k}\sin k\frac{4\pi}{k}=0$$

and

$$x_p'\left(\frac{4\pi}{k}\right)=\frac{1}{2k}\left(\underbrace{\sin k\frac{4\pi}{k}}_{0}+\frac{4\pi}{k}k\underbrace{\cos k\frac{4\pi}{k}}_{1}\right)$$

$$=\frac{2\pi}{k}$$

Problem 4 (cont'd)

$x_p\left(\frac{4\pi}{k}\right)=0$ and $x_p'\left(\frac{4\pi}{k}\right)=\frac{2\pi}{k^2}$ become the initial conditions for the $t \geq \frac{4\pi}{k}$ homogeneous problem. We can write the homogeneous solution as

$$x_h(t) = c_1 \cos k\left(t-\frac{4\pi}{k}\right) + c_2 \sin k\left(t-\frac{4\pi}{k}\right)$$

$$x_h\left(\frac{4\pi}{k}\right) = \underset{\displaystyle 0}{\underset{\parallel}{c_1}}, \qquad x_h'\left(\frac{4\pi}{k}\right) = \underset{\displaystyle \frac{2\pi}{k}}{\underset{\parallel}{kc_2}}$$

So for $t \geq 4\pi/k$,

$$x(t) = \frac{2\pi}{k^2}\sin k\left(t-\frac{4\pi}{k}\right) = \frac{2\pi}{k^2}\sin kt$$

In summary, by this approach,

$$x(t) = \frac{1}{2k}t\sin kt \qquad 0 \leq t \leq \frac{4\pi}{k}$$

$$x(t) = \frac{2\pi}{k^2}\sin kt \qquad \frac{4\pi}{k} \leq t$$

Or, if we use that $k=2$

$$x(t) = \frac{1}{4}t\sin 2t \qquad 0 \leq t \leq 2\pi$$

$$x(t) = \frac{\pi}{2}\sin 2t \qquad 2\pi \leq t$$

Problem 4 (CONT'D) Done with patch method. Back to Laplace transform method.

We got as far as:

$$X(s) = \frac{1}{s^2+k^2}\left[\frac{s}{s^2+k^2} - e^{-4\pi s/k}\frac{s}{s^2+k^2}\right]$$

This pattern comes up so often I just want to deal with it generally:

$$\mathcal{L}^{-1}\left[G(s)\left(1-e^{-as}\right)\right](t) = g(t) - H(t-a)g(t-a)$$

Now I will deal with $\frac{s}{(s^2+k^2)^2}$ by noticing that it is $-\frac{1}{2}\frac{d}{ds}\frac{1}{s^2+k^2}$. But $\frac{1}{s^2+k^2}$ is $\frac{1}{k}\sin kt$ so $\frac{s}{(s^2+k^2)^2}$ is $\frac{1}{2}t\frac{1}{k}\sin kt$.

I think we can put all this into a final answer now:

$$x(t) = \frac{1}{2k}t\sin kt - \frac{1}{2k}H\left(t-\frac{4\pi}{k}\right)\left(t-\frac{4\pi}{k}\right)\underbrace{\sin k\left(t-\frac{4\pi}{k}\right)}_{\sin kt}$$

$$= \frac{1}{2k}t\sin kt + H\left(t-\frac{4\pi}{k}\right)\frac{4\pi}{2k^2}\sin kt - H\left(t-\frac{4\pi}{k}\right)\frac{t\sin kt}{2k}$$

Put in $k=2$ (the original problem):

$$x(t) = \frac{1}{4}t\sin 2t + H(t-2\pi)\frac{\pi}{2}\sin kt - H(t-2\pi)\frac{1}{4}t\sin 2t$$

We have perfect agreement with the patch method.

Plot of Solution to Logan p. 158 #11

In[∘]:=
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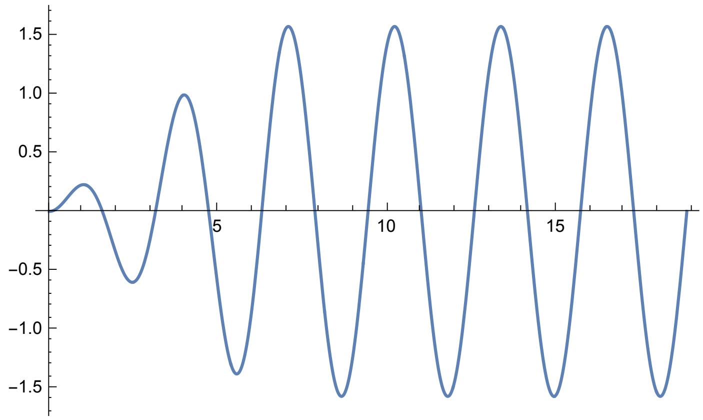
Plot[Piecewise[{{t Sin[2 t] / 4, t ≤ 2 Pi}, {Pi Sin[2 t] / 2, t > 2 Pi}}], {t, 0, 6 Pi}]
```

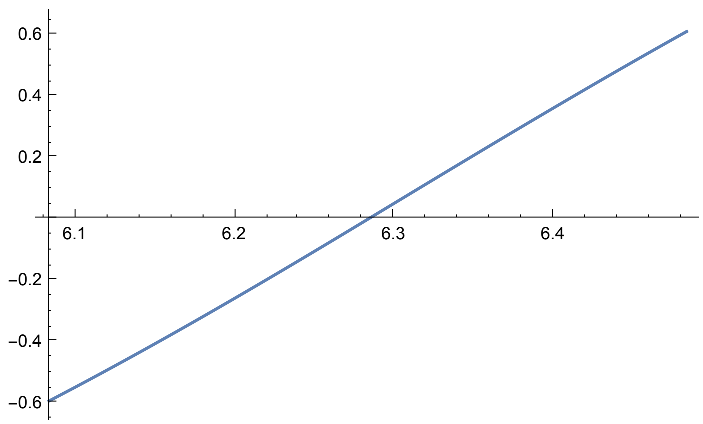
Out[∘]=



Let’s blow up the region at 2π to make sure we have what looks like a smooth join:

In[∘]:=
```
Plot[Piecewise[{{t Sin[2 t] / 4, t ≤ 2 Pi}, {Pi Sin[2 t] / 2, t > 2 Pi}}],
 {t, 2 Pi - 0.2, 2 Pi + 0.2}]
```

Out[∘]=



Problem 5 Logan p. 158 #12

(a) Solve
$$x' = x \quad 0 \le t \le 1$$
with $x(0)=1$
$$x' = x - 2 \quad 1 \le t$$

For $0 \le t \le 1$ that is easy, it is $x(t) = e^t$.

The initial condition for the $t \ge 1$ problem is $x(1) = e$.

The problem $x' - x = -2$ can be solved with an integrating factor.

$$\left(e^{-t} x\right)' = -2e^{-t}$$

So $e^{-t}x = -2\int e^{-t}\,dt = 2e^{-t} + c$

or $x = 2 + ce^t$

Use $x(1) = e$ to get $e = 2 + ce$ or $c = \frac{e-2}{e}$
$= 1 - \frac{2}{e}$

So
$$x(t) = e^t \quad 0 \le t \le 1$$
$$x(t) = 2 + e^t - 2e^{t-1} \quad 1 \le t$$
$$= 2 + e^t(1 - 2/e)$$

Plot of Solution to Logan p. 158 #12 (a)

In[∘]:=
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Plot[Piecewise[{{Exp[t], t ≤ 1}, {2 + Exp[t] (1 - 2 / E), t > 1}}], {t, 0, 2}]
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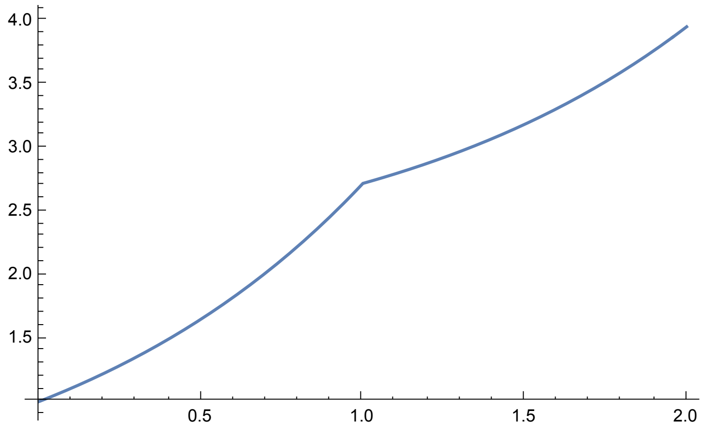
Out[∘]=



Problem 5 (CONT'D)

(b) Solve $x' = x + f(t) \qquad x(0) = 1$

$$f(t) = -2H(t-1)$$

using Laplace transforms

$$sX(s) - \cancel{x(0)}^{\,1} = X(s) + F(s)$$

$$X(s) = \frac{F(s)+1}{s-1}$$

$$F(s) = -2\int_0^\infty H(t-1)e^{-st}\,dt = -\frac{2e^{-s}}{s}$$

So $X(s) = \dfrac{-2e^{-s}/s + 1}{s-1}$

The $\dfrac{1}{s-1}$ term is easy. $\mathcal{L}^{-1}\left[\dfrac{1}{s-1}\right](t) = e^t$

What do we do with $e^{-s}/s\,/\,s-1$?

Partial fractions!

$$\frac{e^{-s}}{s(s-1)} = \frac{e^{-s}}{s-1} - \frac{e^{-s}}{s}$$

$$\mathcal{L}^{-1}\left[e^{-s}g(s)\right](t) \quad \leftarrow \text{my way of writing row 5 with } a=1$$

$$= H(t-1)\,\mathcal{L}^{-1}[g(s)](t-1)$$

with $g(s) = \frac{1}{s-1}$ we get e^{t-1} with $g(s) = \frac{1}{s}$ we get 1

So, to summarize, our answer is

$$x(t) = \underset{\checkmark\checkmark}{e^t} - 2H(t-1)\left[\underset{\checkmark\checkmark\checkmark}{e^{t-1}} - \underset{\checkmark}{1}\right]$$

Obviously this agrees with the patching method for $0 \le t \le 1$.

Let's compare terms for $1 \le t$ (where $H(t-1) = 1$):

$$x(t) = \overset{\checkmark}{2} + \overset{\checkmark\checkmark}{e^t} - \overset{\checkmark\checkmark\checkmark}{2e^{t-1}} \qquad 1 \le t$$

Problem 6 Logan p. 158 #13

$$q'' + q = \begin{cases} t & 0 \le t \le 9 \\ 9 & 9 \le t \end{cases} \quad \text{with } q(0)=0 \text{ and } q'(0)=0$$

$$s^2 Q(s) - s\cancel{q(0)}^{\,0} - \cancel{q'(0)}^{\,0} + Q(s) = F(s)$$

where $f(t) = t - (t-9)\,H(t-9)$

So $F(s) = \dfrac{1}{s^2} - e^{-9s}\dfrac{1}{s^2}$

So $Q(s) = \underbrace{\dfrac{1}{1+s^2}\,\dfrac{1}{s^2}}_{\text{partial fractions}}\left(1 - e^{-9s}\right)$

partial fractions $= \dfrac{1}{s^2} - \dfrac{1}{1+s^2}$

$$Q(s) = \left(\frac{1}{s^2} - \frac{1}{1+s^2}\right)\left(1 - e^{-9s}\right)$$

$$q(t) = t - \sin t - H(t-9)\left[t - 9 - \sin(t-9)\right]$$

Plot of Solution to Logan p. 158 #13

In[2]:=
```
Plot[
 Piecewise[{{t - Sin[t], t ≤ 9}, {t - Sin[t] - (t - 9) + Sin[t - 9], t > 9}}], {t, 0, 20}]
```

Out[2]=