ODE Assignment 11

To present Sunday, June 5

1. Logan p. 158 #15; 2. Logan p. 163 #10

Problem 1 Logan p. 158 #15

Solve — by both the patching method and by the Laplace transform method — the initial value problem

$$x'' + \pi^2 x = \begin{cases} \pi^2, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

where $x(0) = 1$ and $x'(0) = 0$.

I am going to call π "ω" and change the problem to

$$x'' + \omega^2 x = \begin{cases} \omega^2, & 0 < t < \frac{\pi}{\omega} \\ 0, & t > \frac{\pi}{\omega} \end{cases}$$

It is of course the same problem if you stick in $\omega = \pi$.

Problem 1 (CONT'D)

Solution by patching method.

From 0 to $\frac{\pi}{\omega}$ we have

$$x'' + \omega^2 x = \omega^2$$

We need a particular solution of the homogeneous equation. I can just see it: it is $x_p(t) \equiv 1$. Also, that solution satisfies $x_p(0) = 1$ and $x_p'(0) = 0$, so we don't have to add in any of the homogeneous sol'n.

At $t = \frac{\pi}{\omega}$, the forcing function is suddenly removed. The system will now oscillate with some combination of

$$c_1 \cos \omega t + c_2 \sin \omega t$$

For this period, the initial condition is

$$x\left(\frac{\pi}{\omega}\right) = 1 \qquad \text{and} \qquad x'\left(\frac{\pi}{\omega}\right) = 0$$

Plugging $\frac{\pi}{\omega}$ in for t, we have

$$c_1 \underbrace{\cos \cancel{\omega} \frac{\pi}{\cancel{\omega}}}_{-1} + c_2 \underbrace{\sin \cancel{\omega} \frac{\pi}{\cancel{\omega}}}_{0} = c_1 = 1$$

Doing the same thing with the first derivative, we have

$$-c_1 \omega \underbrace{\sin \cancel{\omega} \frac{\pi}{\cancel{\omega}}}_{0} + c_2 \omega \underbrace{\cos \cancel{\omega} \frac{\pi}{\cancel{\omega}}}_{-1} = -c_2 = 0$$

So we are done and can summarize

$$x(t) = \begin{cases} 1 & 0 \le t \le \frac{\pi}{\omega} \\ -\cos \omega t & \frac{\pi}{\omega} < t \end{cases}$$

Problem 1 (CONT'D)

Solution by Laplace transform method. We rewrite the probem using the Heaviside function.

$$x'' + \omega^2 x = \omega^2 - \omega^2 H\left(t - \frac{\pi}{\omega}\right)$$

We Laplace transform the equation

$$s^2 X(s) - s\,\cancelto{1}{x(0)} - \cancelto{0}{x'(0)} + \omega^2 X(s) = \omega^2 \frac{1}{s}\left(1 - e^{-\frac{\pi}{\omega}s}\right)$$

So

$$X(s) = \frac{1}{s^2+\omega^2}\left[\underset{\uparrow\ \text{1st term}}{s} + \frac{\omega^2}{s}\left(\underset{\uparrow\ \text{2nd term}}{1} - \underset{\uparrow\ \text{3rd term}}{e^{-\frac{\pi}{\omega}s}}\right)\right]$$

Consulting the table on p. 175, the first term is $\cos \omega t$. The second term is the product of $\frac{1}{s^2+\omega^2}$ and $\frac{\omega^2}{s}$. $\frac{1}{s^2+\omega^2}$ is $\frac{1}{\omega}\sin \omega t$.

$\frac{\omega^2}{s}$ is $\omega^2 H(t)$. The product rule near the bottom of p. 175, tells us

$$\int_0^t \frac{1}{\cancel{\omega}} \sin \omega\tau \cdot \omega^{\cancel{2}}\, \overbrace{H(t-\tau)}^{1}\, d\tau = -\cos \omega t + 1$$

Problem 1 (CONT'D)

So the 1st and 2nd terms together have just given us 1. The third term is $-e^{-\frac{\pi}{\omega}s}$ times the second term. The second term gave us

$$-\cos \omega t + 1$$

The third term therefore gives us

$$-H\left(t-\frac{\pi}{\omega}\right)\left[-\cos \omega\left(t+\frac{\pi}{\omega}\right)+1\right]$$

$$= H\left(t-\frac{\pi}{\omega}\right)\left[-\cos \omega t - 1\right]$$

To summarize

$$x(t) = \underbrace{1}_{\text{from 1st and 2nd terms}} + \underbrace{H\left(t-\frac{\pi}{\omega}\right)\left[-\cos\omega t - 1\right]}_{\text{from 3rd term}}$$

This is in perfect agreement with the patching method.

Problem 2 Logan p. 163 #10

Write an integral expression for the inverse transform of

$$X(s) = \frac{1}{s} e^{-3s} F(s)$$

We notice this is of the form $F(s)G(s)$ where $G(s) = \frac{1}{s}e^{-3s}$

Oh, so $g(t) = H(t-3)$.

Now we use the convolution property

$$\mathcal{L}^{-1}[F(s)G(s)](t)$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t f(t) \underbrace{H(t-\tau-3)}_{\text{rewrite very slightly as}} d\tau$$

$H(t-3-\tau) \leftarrow$ if $\tau < t-3$ this is 1, if $\tau > t-3$ this is 0

$$= \int_0^{t-3} f(t)\, dt$$

(provided $t > 3$ — if $t \leq 3$ this is just 0)