ODE Assignment 12

To turn in Tuesday, June 7

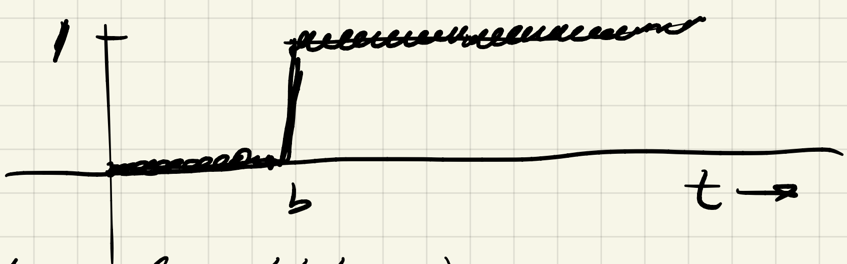
1. Trailing Averages; 2. p.173 #4; 3. p.174 #6; 4. p.174 #8

Problem 1 Trailing Averages

assume $b<a$ and $a<t$

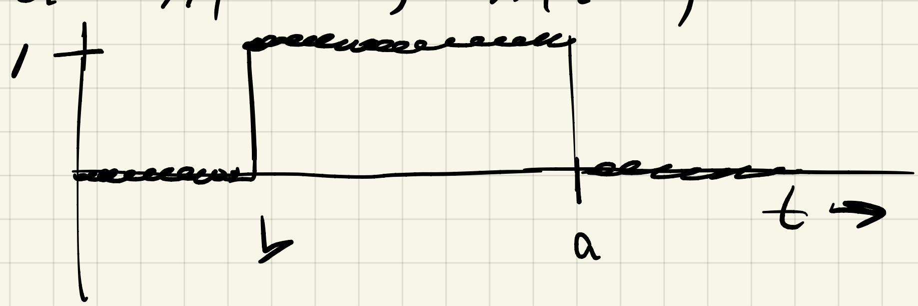
(a) The difference of Heaviside functions that is only non-zero for $b<t<a$ is $H(t-b)-H(t-a)$

Graph of $H(t-b)$



Graph of $H(t-a)$

Graph of $H(t-b)-H(t-a)$



Problem 1 (CONT'D)

(b) Let $g(t) = H(t-b) - H(t-a)$. What is $G(s)$?

used the definition of the Laplace transform and then the main property of $g(t)$—which is that it is nonzero only from b to a

$$G(s) = \int_0^\infty g(t)e^{-st}\,dt = \int_b^a e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_b^a = \frac{1}{s}\left(e^{-bs} - e^{-as}\right)$$

$$\mathcal{L}^{-1}[G(s)](t) = H(t-b) - H(t-a) \checkmark\checkmark$$

used the table on p.175 (line 4) and the double-check is trivial

(c) $$\frac{1}{a-b}\int_{t-a}^{t-b} f(\tau)\,d\tau$$

$$= \frac{1}{a-b}\int_0^t f(\tau)\underbrace{g(t-\tau)}\,d\tau$$

because $g(t-\tau)$ is only nonzero when

$b < t-\tau < a$

or $b - t < -\tau < a - t$

or $t - b > \tau > t - a$

these are important steps

THESE ASSUMPTIONS ARE ESSENTIAL

assume $b < a$ and $a < t$

(d) By the convolution property (p. 160), the Laplace transform of this is

$$\frac{1}{a-b}F(s)\frac{1}{s}\left(e^{-bs} - e^{-as}\right)$$

If we put $b = 0$, then this is

so the end of the window is t

$$\frac{1}{a}F(s)\frac{1}{s}\left(1 - e^{-as}\right) \leftarrow$$

which is what I originally thought Logan was going after in p.163 #10

Problem 2 Logan p. 173 #4

Instead of $\delta_2(t)$, I prefer $\delta(t-2)$.

Solve

$$x'' + x = \delta(t-2) \quad \text{with } x(0) = x'(0) = 0$$

$$\underbrace{s^2 X(s) - s\,x(0) - x'(0)}_{\substack{\text{Theorem 3.11, p. 147} \\ \text{Equation 3.7}}} + X(s) = \underbrace{e^{-2s}}_{\substack{\text{what Logan calls} \\ \text{"the sifting property"} \\ \text{on p. 168}}}$$

So $X(s) = \dfrac{e^{-2s}}{1+s^2}$

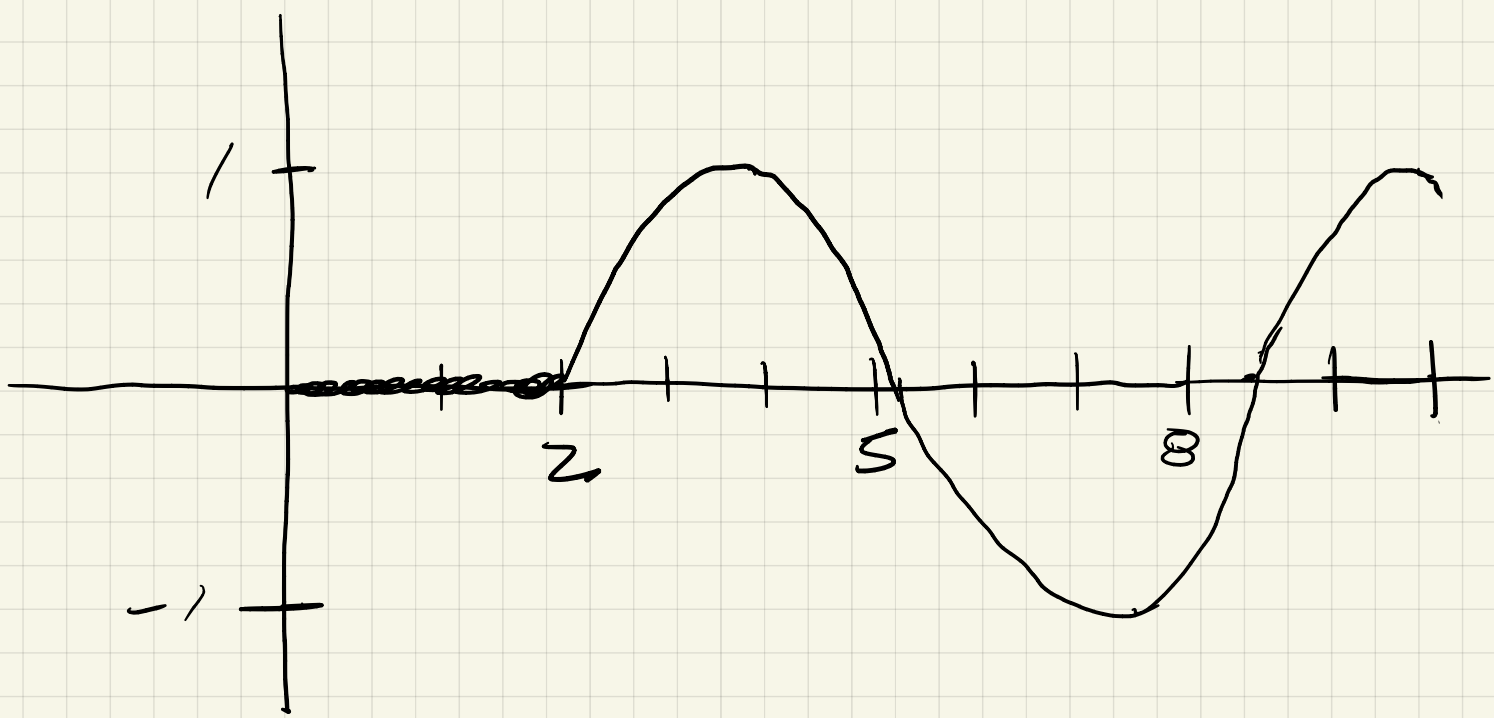
The inverse Laplace transform of $\dfrac{1}{1+s^2}$ is $\sin t$

By line 5 of the table $e^{-as}F(s)$ becomes $H(t-a)f(t-a)$

So $\dfrac{e^{-2s}}{1+s^2}$ becomes

$$H(t-2)\sin(t-2)$$

Sketch it:



Problem 3 Logan p. 174 #6

$$x'' + 4x = \delta(t-2\pi) - \delta(t-5\pi)$$

with $x(0)=0$, $x'(0)=0$

Laplace transform of LHS is

$$\frac{1}{s^2+4}$$

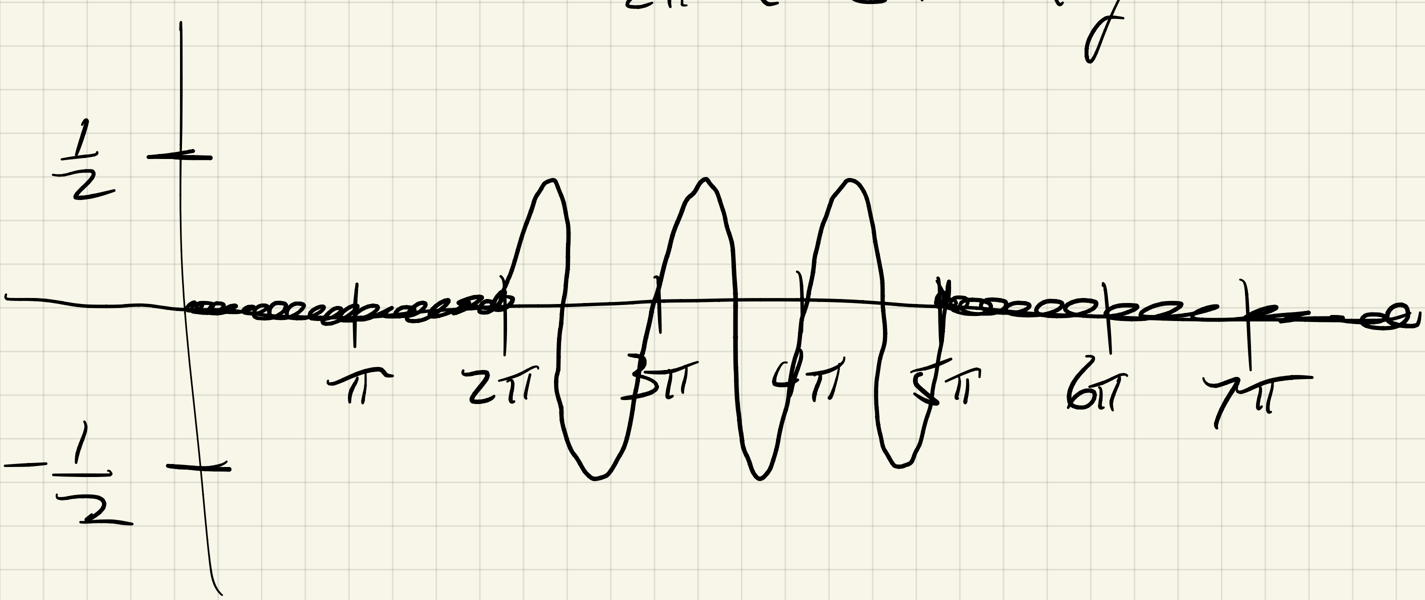
Laplace transform of RHS is

$$e^{-2\pi s} - e^{-5\pi s}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\left(e^{-2\pi s} - e^{-5\pi s}\right)\right](t)$$

$$= \frac{1}{2}\left[H(t-2\pi)\sin 2(t-2\pi) - H(t-5\pi)\sin 2(t-5\pi)\right]$$

$$= \frac{1}{2}\underbrace{\left(H(t-2\pi) - H(t-5\pi)\right)}_{\text{Nonzero from } 2\pi \text{ to } 5\pi \text{ only}} \sin 2t$$



Problem 4 Logan p. 174 #8

$$x'' + x = f(t) \quad \text{with } x(0)=0,\ x'(0)=0$$

and $f(t) = \delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \cdots$

If $f(t)$ were just one of these, we know the answer:

$\delta(t - m\pi)$ becomes $e^{-m\pi s}$

and $x(t)$ is the inverse transform of

$$\frac{e^{-m\pi s}}{1+s^2} = H(t-m\pi)\sin(t-m\pi)$$

$\sin(t-m\pi)$ is $\sin t$ if m is even and $-\sin t$ if m is odd.

Since we have an infinite series of unit impulses, the response is the sum

$$x(t) = \sum_{m=0}^{\infty} (-1)^m H(t-m\pi)\sin t$$

Here is the graph: