ODE Assignment 13

To present Thursday, June 9

1. p.190 #1 ; 2. p.190 #2 ; 3. p.190 #3

For all three problems, do parts (a) and (c) only.

Problem 1. Logan p.190 #1

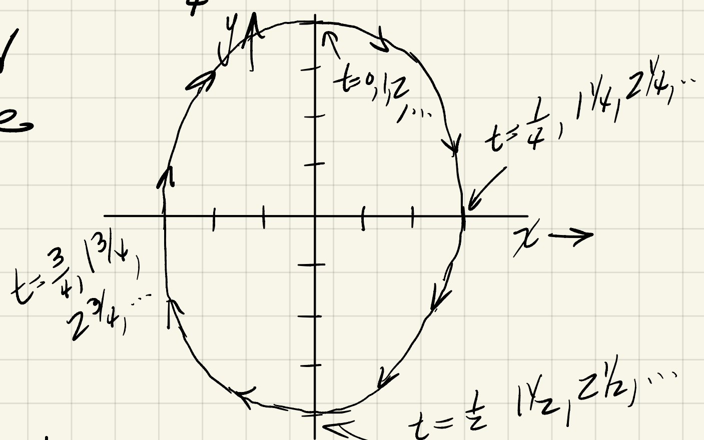
(a) $x = 3\sin 2\pi t \qquad y = 4\cos 2\pi t$

What curve in the x–y plane is traced out by this parametrization?

Since $\frac{x}{3} = \sin 2\pi t$ and $\frac{y}{4} = \cos 2\pi t$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

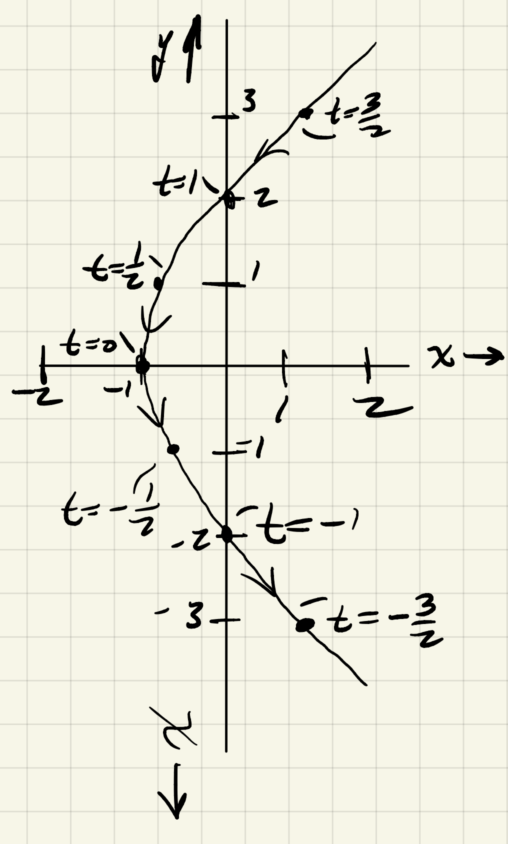
This is an ellipse of width 3 and height 4.



(c) $x = t^2 - 1, \quad y = 2t$ so $t = \frac{y}{2}$

So $x = \left(\frac{y}{2}\right)^2 - 1$

This is a parabola opening to the right



Problem 2 Logan p. 190 #2

(a) $x' = -3y, \quad y' = 2x$

Divide equations (see p. 186):

$$\frac{y'}{x'} = -\frac{2x}{3y} \qquad \text{use} \quad \frac{y'}{x'} = \frac{dy}{dx}$$

a consequence of the chain rule

$\left(\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}\right.$ and $\left.\frac{dt}{dx} = \frac{1}{x'}\right)$

$$\frac{dy}{dx} = -\frac{2}{3}\,\frac{x}{y}$$

$$\int y\,dy = -\frac{2}{3}\int x\,dx$$

$$\frac{y^2}{2} = -\frac{2}{3}\,\frac{x^2}{2} + c$$

$$3y^2 = -2x^2 + d$$

$$3y^2 + 2x^2 = d$$ ← clearly d must be positive (or 0)

$$\frac{3}{2}y^2 + x^2 = w^2$$ ← let's call it $2w^2$

These are ellipses of width $2w$. Their height is $2\sqrt{\frac{2}{3}}\,w$. In other words they are a little wider than they are tall.

Problem 2 (CONT'D) This is still part (a).

$\frac{3}{2}y^2 + x^2 = w^2$ is our phase-space sol'n.

Let us take $\frac{d}{dt}$ of it to double-check.

$$3y\underbrace{\frac{dy}{dt}}_{2x} + 2x\underbrace{\frac{dx}{dt}}_{-3y} = 0 \quad \checkmark\checkmark$$

Logan also asks us to find "the general solution" in component form and in vector form (see p. 185).

From $x' = 3y$ we have $x'' = 3y'$. Put that into $y' = 2x$ to get

$$x'' = 3 \cdot 2x = 6x.$$

The sol'n to that is

$$x = c_1 \cos\sqrt{6}\,t + c_2 \sin\sqrt{6}\,t$$

Using $y = \frac{-1}{3}x'$, we have

$$y = -\frac{c_2\sqrt{6}}{3}\cos\sqrt{6}\,t + \frac{c_1\sqrt{6}}{3}\sin\sqrt{6}\,t$$

In vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} \cos\sqrt{6}\,t \\ \frac{\sqrt{6}}{3}\sin\sqrt{6}\,t \end{pmatrix} + c_2\begin{pmatrix} \sin\sqrt{6}\,t \\ -\frac{\sqrt{6}}{3}\cos\sqrt{6}\,t \end{pmatrix}$$

Problem 2 (CONT'D) Onward to part (c).

(c) $x' = -3x \quad y' = 2y$

This is interesting and different. There is no coupling between y and x. We can still divide equations as instructed:

$$\frac{y'}{x'} = -\frac{2y}{3x} = \frac{dy}{dx}$$

$$\int \frac{dy}{y} = -\frac{2}{3}\int \frac{dx}{x} \quad \text{or} \quad \ln y = -\frac{2}{3}\ln x + c$$

Or $\quad y = d\, x^{-\frac{2}{3}} \quad$ (where $d = e^c$)

Or $\quad y^3 = e\, x^{-2} \quad$ (where $e = d^3$)

Or $\quad y^3 = \frac{y_0^3}{x_0^2} x^{-2}$

I am going to let Mathematica make me a bunch of nice examples of our phase-space sol'ns.

Since there is no coupling between x and y we can also just solve

$x' = -3x, \quad y' = 2y \quad$ independently.

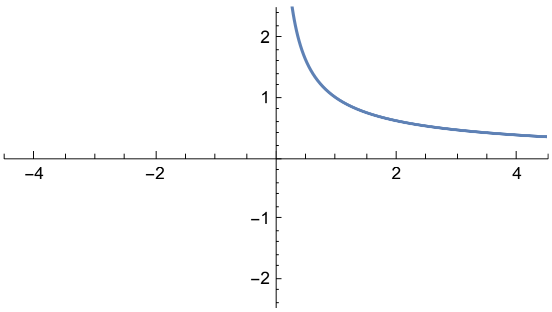
$x = x_0 e^{-3t}, \quad y = y_0 e^{2t}$

In vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = x_0 \begin{pmatrix} e^{-3t} \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} = \begin{pmatrix} x_0 e^{-3t} \\ y_0 e^{2t} \end{pmatrix}$$

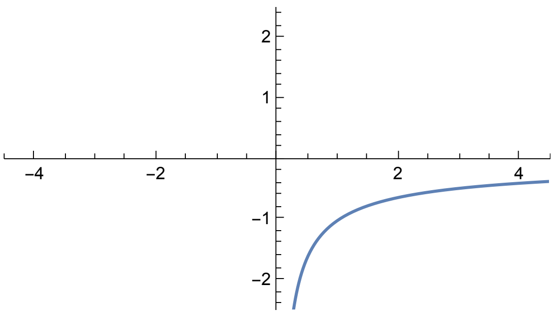
In[45]:=
```
curves[t_] := {x0 Exp[-3 t], y0 Exp[2 t]}; range = {{-4.5, 4.5}, {-2.5, 2.5}};
ParametricPlot[curves[t] /. {x0 → 1, y0 → 1}, {t, -0.5, 0.5}, PlotRange → range]
```

Out[46]=



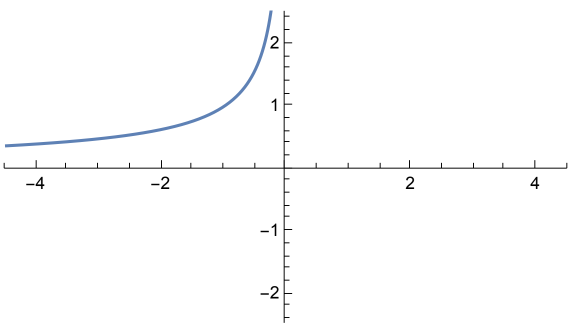
In[47]:=
```
ParametricPlot[curves[t] /. {x0 → 1, y0 → -1}, {t, -0.5, 0.5}, PlotRange → range]
```

Out[47]=



In[48]:=
```
ParametricPlot[curves[t] /. {x0 → -1, y0 → 1}, {t, -0.5, 0.5}, PlotRange → range]
```

Out[48]=



In[49]:=
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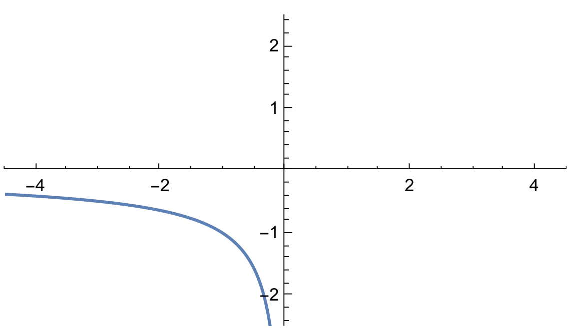
ParametricPlot[curves[t] /. {x0 → -1, y0 → -1}, {t, -0.5, 0.5}, PlotRange → range]
```

Out[49]=



Problem 3 Logan p. 190 #3

(a) Apply method of elimination to

$$x' = x, \quad y' = x + 2y$$

Because the first equation does not involve y, we can just write the solution:

$$x = x_0 e^t$$

Now we stick that into the second eqn.:

$$y' = x_0 e^t + 2y$$

We uses the integrating factor method

$$y' - 2y = x_0 e^t$$

$$(e^{-2t} y)' = e^{-2t} x_0 e^t = x_0 e^{-t}$$

$$e^{-2t} y = -x_0 e^{-t} + c$$

$$y = e^{2t}(-x_0 e^{-t} + c)$$

Put in $t=0$ and get $y_0 = -x_0 + c$ so $c = x_0 + y_0$

$$y = e^{2t}(-x_0 e^{-t} + x_0 + y_0)$$

In vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = x_0 \begin{pmatrix} e^t \\ -e^t + e^{2t} \end{pmatrix} + y_0 \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

Problem 3 (CONT'D)

It is wise to double-check part (a).
We got:

$$\begin{pmatrix} x \\ y \end{pmatrix} = x_0 \begin{pmatrix} e^t \\ -e^t+e^{2t} \end{pmatrix} + y_0 \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

Take $\frac{d}{dt}$ of both sides

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = x_0 \begin{pmatrix} e^t \\ -e^t+2e^{2t} \end{pmatrix} + y_0 \begin{pmatrix} 0 \\ 2e^{2t} \end{pmatrix}$$

Or
$$x' = x_0 e^t \checkmark\checkmark$$
$$y' = x_0(-e^t+2e^{2t}) + 2y_0 e^{2t}$$

Is the RHS of the eqn. for y' the same as $x+2y$?

$$x+2y = x_0 e^t + 2\left(x_0(-e^t+e^{2t}) + y_0 e^{2t}\right)$$
$$= -x_0 e^t + 2x_0 e^{2t} + 2y_0 e^{2t} \checkmark\checkmark$$

Problem 3. On to part (c)

(c) $x' = x + 2y \qquad y' = x$

$x'' = x' + 2y' \qquad y'' = x'$

$$y'' = x' = x + 2y = y' + 2y$$

So $y'' - 2y - y' = 0$

$\lambda^2 - \lambda - 2 = 0 \quad$ or $\quad (\lambda - 2)(\lambda + 1) = 0$

So $\quad y = c_1 e^{2t} + c_2 e^{-t}$

And

$$x = y' = 2c_1 e^{2t} - c_2 e^{-t}$$

In vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

Problem 3 (CONT'D). It is wise to double-check part (c). We got:

$$y = c_1 e^{2t} + c_2 e^{-t}$$

$$x = 2c_1 e^{2t} - c_2 e^{-t}$$

Clearly $y' = x$ is satisfied. ✓✓

How about $x' = x + 2y$? By differentiating

$$x' = 4c_1 e^{2t} + c_2 e^{-t}$$

Is this the same as

$x + 2y$?

$$x + 2y = 2c_1 e^{2t} - c_2 e^{-t} + 2(c_1 e^{2t} + c_2 e^{-t})$$

$$= 4c_1 e^{2t} + c_2 e^{-t}$$

✓✓ ✓✓