ODE Assignment 14

To turn in Sunday, June 12

1. p.198 #1; 2. p. 198 #2; 3. p. 201 #3; 4. p. 207 #4; 5. p. 207 #5

Problem 1 Logan p. 198 #1

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} \qquad x = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 5 & 11 \end{pmatrix}$$

$$B - 4A = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} - \begin{pmatrix} 4 & 12 \\ 8 & 16 \end{pmatrix} = \begin{pmatrix} -5 & -12 \\ -5 & -9 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 8 & 21 \\ 10 & 28 \end{pmatrix}$$

$$BA = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 17 & 37 \end{pmatrix}$$

$\det A = -2 \qquad \det B = -7$

$\det AB = 224 - 210 = 14$ ✓✓

$\det BA = -37 + 51 = 14$ ✓✓

I caught two mistakes this way. Acck!

$$A^2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 10 & 22 \end{pmatrix}$$

$\det A^2 = 154 - 150 = 4$ ✓✓

Problem 1 (CONT'D)

$$Bx = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ -29 \end{pmatrix}$$

$$ABx = \begin{pmatrix} 8 & 21 \\ 10 & 28 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 16-105 \\ 20-140 \end{pmatrix} = \begin{pmatrix} -89 \\ -120 \end{pmatrix}$$

$$A^{-1} = \frac{\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}}{-2} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

Let's check

$$A^{-1}A = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark\checkmark$$

$$B^3 = B\,B^2 = B\begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 18 & 49 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 129 & 343 \end{pmatrix}$$

$$\det B^3 = -343 \checkmark\checkmark$$

$$AI = A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$B^{-1} = \frac{\begin{pmatrix} 7 & 0 \\ -3 & -1 \end{pmatrix}}{-7} = \begin{pmatrix} -1 & 0 \\ \frac{3}{7} & \frac{1}{7} \end{pmatrix}$$

Problem 2 Logan p. 198 #2

Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $\qquad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(a) Solve $Ax = b$

$$x = A^{-1} b = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ \frac{3}{2} \end{pmatrix}$$

Let's check

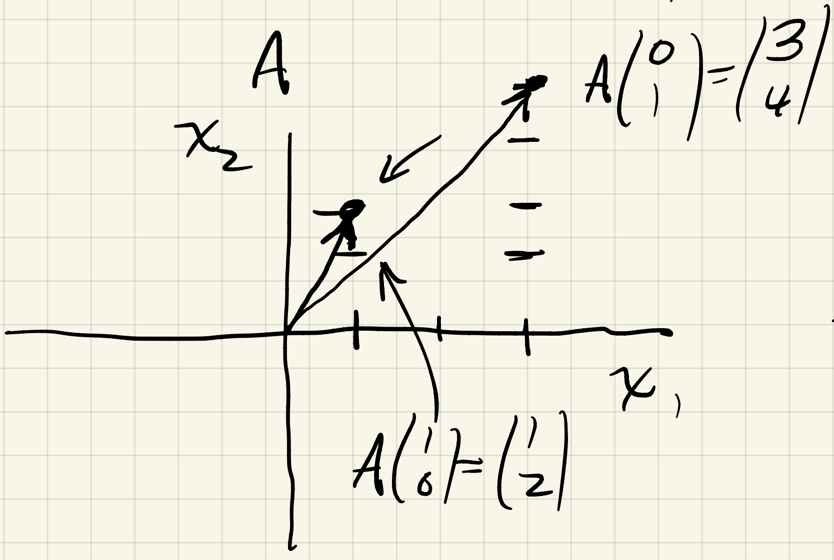
$$Ax = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -\frac{5}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} + \frac{9}{2} \\ -5 + 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \checkmark\checkmark$$

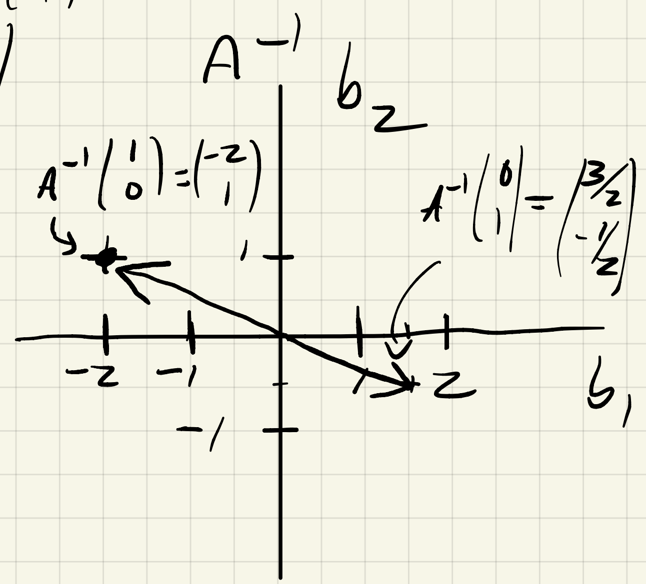
(b) Same thing by Cramer's rule

$$x_1 = \frac{\det \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det A} = \frac{\det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}}{-2} = -\frac{5}{2}$$

$$x_2 = \frac{\det \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det A} = \frac{\det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}{-2} = \frac{3}{2}$$

(c) Not sure what geometric illustration Logan is looking for here but here is what A and A^{-1} do to the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$





Problem 3 Logan p. 201 #3

(a) Find the critical point of

$$x' = 2x + 3y$$
$$y' = -x - 14$$

and then transform the system into a homogeneous system. "Critical point" is jargon for time-invariant solution. E.g. $x' = y' = 0 \Longrightarrow$

$$0 = 2x^* + 3y^*$$
$$0 = -x^* - 14$$

in matrix form $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \overset{A}{\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} x^* \\ y^* \end{pmatrix} + \begin{pmatrix} 0 \\ -14 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \frac{\begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}}{3} \begin{pmatrix} 0 \\ 14 \end{pmatrix} = \begin{pmatrix} -\frac{42}{3} \\ \frac{28}{3} \end{pmatrix}$$

By creating new variables v and w

$$v = x - x^* = x + 14 \qquad w = y - y^* = y - \frac{28}{3}$$
$$v' = x' \qquad w' = y'$$

we then have

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = A \begin{pmatrix} v \\ w \end{pmatrix} \quad \text{where } A \text{ is still } \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$$

without the matrix and vector notation this is

$$v' = 2v + 3w$$

$$w' = -v$$

Problem 3 (CONT'D)

(b) First, find the critical point of

$$x' = -x + 3y - 6$$
$$y' = x + 2y - 1$$

In matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -6 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A\begin{pmatrix} x^* \\ y^* \end{pmatrix} + \begin{pmatrix} -6 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{12}{5}+\frac{3}{5} \\ \frac{6}{5}+\frac{1}{5} \end{pmatrix} = \begin{pmatrix} -\frac{9}{5} \\ \frac{7}{5} \end{pmatrix}$$

As in part (a), we introduce r and w

$$v = x - x^* = x + \frac{9}{5} \qquad w = y - y^* = y - \frac{7}{5}$$
$$v' = x' \qquad\qquad w' = y'$$

We then have

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = A\begin{pmatrix} v \\ w \end{pmatrix} \quad \text{where } A \text{ is still } \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$$

Without the matrix and vector notation, this is

$$v' = -v + 3w$$
$$w' = v + 2w$$

Problem 4 Logan p. 207 #4

A proof. Assume λ is a non-zero eigenvalue of a matrix A. In equations, this means there is an eigenvector x such that

$$Ax = \lambda x$$

the identity matrix

We know that $A^{-1}A = \mathbb{1}$. To use that, multiply both sides of the above equation by A^{-1}.

$$A^{-1}Ax = \lambda A^{-1}x$$

$$A^{-1}Ax = \mathbb{1}x = x \qquad \text{So} \qquad x = \lambda A^{-1}x.$$

Now multiply both sides by λ^{-1} (which we can do because it was assumed that $\lambda \neq 0$.

$$\lambda^{-1}x = A^{-1}x$$

This says that x is an eigenvector of A^{-1} with eigenvalue λ^{-1} ∎

Problem 5 Logan p. 207 #5

Another proof. Assume λ is an eigenvalue of a matrix A. In equations, this means there is an eigenvector x such that

$$Ax = \lambda x$$

Multiply both sides of this equation by A^{n-1}. We have

$$A^n x = \lambda x$$

But

$$A^n x = \overbrace{A \cdot \cdot \cdot \cdot \; A}^{n \text{ times}} x$$

and when each A acts on x, it becomes λx. This happens n times, so we get $\lambda^n x$.

A stickler might prefer induction.

Also, at some point, both in this proof and the preceding proof, we are using associativity of matrix multiplication. E.g., we are using things like

$$A^3 x = (A \cdot A \cdot A) x = (A \cdot A) A x.$$