ODE Assignment 15

Assignment 15 to present on Tuesday, June 14: 1. p. 207 #1; 2. p. 218 #2; 3. p. 221 #2; 4. p. 225 #1 — NB: (1) To cut down on the total amount of work, on all the multi-part problems, do parts (a) and (b) only; (2) Also, in Problems 3 and 4, just find the general solutions (we will do the phase diagrams that Logan requested as part of the next assignment)

Problem 1 Logan p. 207 #1

(a) Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$.

Set
$$\det \begin{pmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)^2 - 4 = 0 \qquad \lambda = 2 \pm 2$$

e.g. $\lambda_+ = 4, \quad \lambda_- = 0$

For λ_+:
$$\begin{pmatrix} 2-4 & -1 \\ -4 & 2-4 \end{pmatrix}\begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or
$$\begin{pmatrix} +2 & +1 \\ +4 & +2 \end{pmatrix}\begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $2x_+ + y_+ = 0$

If $x_+ = +1$ then $y_+ = -2$.

For λ_-:
$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x_- \\ y_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $2x_- - y_- = 0$

If $x_- = +1$ then $y_- = 2$.

Problem 1 (CONT'D)

(b) Same thing for $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

Set
$$\det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0 \qquad \lambda = 1 \pm 2$$

e.g. $\lambda_+ = 3$, $\lambda_- = -1$

For λ_+:
$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or
$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or
$$-2x_+ + y_+ = 0$$

If $x_+ = 1$ then $y_+ = 2$.

For λ_-:
$$\begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix}\begin{pmatrix} x_- \\ y_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or
$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_- \\ y_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or
$$2x_- + y_- = 0$$

If $x_- = 1$ then $y_- = -2$.

There are similarities worth contemplating between parts (a) and (b). For example, the spacing between λ_+ and λ_- was 4 in both parts.

Problem 2 Logan p. 218 #2

(a) Solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Assume $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$

Eigenvalue equation is then

$$\det \begin{pmatrix} -1-\lambda & 1 \\ 0 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Or $(-1-\lambda)(-3-\lambda) = 0$

So $\lambda_1 = -1, \quad \lambda_2 = -3.$

For λ_1: $\begin{pmatrix} -1-(-1) & 1 \\ 0 & -3-(-1) \end{pmatrix} \begin{pmatrix} x_{01} \\ y_{01} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_{01} \\ y_{01} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Or $y_{01} = 0$ and x_{01} can be anything, so we'll choose $x_{01} = 1$

For λ_2: $\begin{pmatrix} -1-(-3) & 1 \\ 0 & -3-(-3) \end{pmatrix} \begin{pmatrix} x_{02} \\ y_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{02} \\ y_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $2x_{02} + y_{02} = 0$. If we choose $x_{02} = 1$, then $y_{02} = -2$.

Let us summarize: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}.$

Problem 2 (CONT'D)

(b) Solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Assume $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$

Eigenvalue equation is then

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 3 & -4-\lambda \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Or $(1-\lambda)(-4-\lambda) = 0$

So $\lambda_1 = 1$ and $\lambda_2 = -4$.

For λ_1:

$$\begin{pmatrix} 0 & 0 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x_{01} \\ y_{01} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$3x_{01} - 5y_{01} = 0$$

If $x_{01} = 1$ then $y_{01} = \frac{3}{5}$.

For λ_2:

$$\begin{pmatrix} 5 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_{02} \\ y_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $5x_{02} = 0$. So $x_{02} = 0$ and we can choose y_{02} to be anything so we'll choose $y_{02} = 1$.

Let us summarize: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3/5 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-4t}$.

Problem 3 Logan p. 221 #2

Just find the general solution:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

← phase diagram on next assignment

Eigenvalue equation is

$$\det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)+1 = 0$$

or $\lambda^2 - 3\lambda + 3 = 0$

$$\lambda_\pm = \frac{3 \pm \sqrt{9-12}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}\, i}{2}$$

For λ_+:

$$\begin{pmatrix} 1 - \frac{3}{2} - \frac{\sqrt{3}\, i}{2} & -1 \\ 1 & 2 - \frac{3}{2} - \frac{\sqrt{3}\, i}{2} \end{pmatrix} \begin{pmatrix} x_{o+} \\ y_{o+} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}\, i}{2} & -1 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}\, i}{2} \end{pmatrix} \begin{pmatrix} x_{o+} \\ y_{o+} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiply the bottom equation

$$1 \cdot x_{o+} + \left(\frac{1}{2} - \frac{\sqrt{3}\, i}{2}\right) \cdot y_{o+} = 0$$

by $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$. It becomes

$$\left(-\frac{1}{2} - \frac{\sqrt{3}\, i}{2}\right) \cdot x_{o+} + \underbrace{\left(-\frac{1}{4} + \frac{1}{4}\sqrt{3}\, i - \frac{1}{4}\sqrt{3}\, i - \frac{3}{4}\right)}_{-1} y_{o+} = 0.$$

So that is indeed the top equation. Choose $x_{o+} = 1$. By the top equation $y_{o+} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

Problem 3 (CONT'D)

We have found one solution

$$\begin{pmatrix} 1 \\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i \end{pmatrix} e^{\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)t} = \begin{pmatrix} 1 \\ -\frac{1}{2}-\frac{\sqrt{3}i}{2} \end{pmatrix} e^{\frac{3}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right)$$

We could grind out another solution but starting with λ_- (which is λ_+^*), but because we started with a differential equation with real coefficients, on general grounds we know that a second solution is the complex conjugate of this solution. E.g.,

$$\begin{pmatrix} 1 \\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i \end{pmatrix} e^{\frac{3}{2}t}\left(\cos\frac{\sqrt{3}}{2}t - i\sin\frac{\sqrt{3}}{2}t\right)$$

The sum of these two solutions divided by 2 is

$$\begin{pmatrix} \cos\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix} e^{\frac{3}{2}t}$$

The difference divided by $2i$ is

$$\begin{pmatrix} \sin\frac{\sqrt{3}}{2}t \\ -\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix} e^{\frac{3}{2}t}$$

The most general real solution is any real linear combination of these two solutions.

Problem 4 p. 225 #1

(a) Eigenvalue equation is:

$$\det\begin{pmatrix} -3-\lambda & 1 \\ 0 & -3-\lambda \end{pmatrix} = (-3-\lambda)^2 = 0$$

Double root with $\lambda = -3$. If we use

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{-3t}$$ as a solution, we must have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ The upper equation tells us $y_0 = 0$. The lower equation tells us nothing.

So one solution is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t}$$

The second solution is of the form

$$\begin{pmatrix} x_0 + v_0 t \\ y_0 + w_0 t \end{pmatrix} e^{-3t}$$

By subtracting x_0 times the first solution we can simplify this to

$$\begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} e^{-3t}$$ (where $r_0 = y_0 - x_0$)

Problem 4 (CONT'D)

We need see what putting $\begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} e^{-3t}$ into

$$\left(\begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} e^{-3t} \right)' = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} e^{-3t}$$

tells us about v_0, r_0, and w_0

When the derivative on the left-hand side hits the e^{-3t}, that gives us

$$\begin{pmatrix} v_0 t \\ r_0 + v_0 t \end{pmatrix} (-3) e^{-3t}$$

which exactly cancels out the diagonal elements on the right-hand side. So we are left with

$$\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \cancel{e^{-3t}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} \cancel{e^{-3t}}$$

The lower equation tells us $w_0 = 0$.
The upper equation is then just

$$v_0 = r_0$$

We can choose $v_0 = r_0 = 1$. So our other solution is $\begin{pmatrix} t \\ 1 \end{pmatrix} e^{-3t}$ and the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix} e^{-3t}$$

Problem 4 (CONT'D)

(b) Eigenvalue equation is:

$$\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)+1=0$$

$$\lambda^2-4\lambda+4=0 \quad \text{or} \quad (\lambda-2)^2=0 \quad \text{or} \quad \lambda=2.$$

So although it was not obvious looking at the original matrix, this is another double-root situation and one solution is

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{2t} \text{ with } \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$$

So if we choose $x_0=1$, then $y_0=-1$.

The other solution is of the form

$$\begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} e^{2t}$$

When the derivative on the left-hand side of the original equation hits e^{2t} we get...

$$2\begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} e^{2t}$$

when the derivative hits the t in $v_0 t$ or $w_0 t$, we get...

$$\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} e^{2t}$$

In total (after cancelling the e^{2t} everywhere) we have...

$$2\begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix} + \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_0 t \\ r_0 + w_0 t \end{pmatrix}$$

Problem 4 (CONT'D)

By the way, the lower equation tells us the same thing.

The upper equation is

$$\cancel{2}v_0 t + v_0 = v_0 \cancel{t} - r_0 - w_0 t$$

From this, we learn $v_0 = -w_0$ and $v_0 = -r_0$

So if we choose $v_0 = 1$, then $w_0 = -1$ and $r_0 = -1$.

Let us summarize. The most general solution is

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t \\ -1-t \end{pmatrix} e^{2t}$$