ODE Assignment 16

Assignment 16 to turn in Thursday, June 16: 1. p. 219 #3 parts (a) and (b) only; 2. Do the diagrams requested in p. 221 #2 (the general solution was found in the previous assignment); 3. Do the diagrams requested in p. 225 #1 (the general solution for this was also found in the previous assignment); 4. p. 239 #13 (this is the only problem with an application that Logan offered in Section 4.5 other than electrical circuits)

Problem 1 Logan p. 219 #3

(a) First, an analytical approach:

We found: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}$

We have

$$x = c_1 e^{-t} + c_2 e^{-3t}$$

$$y = -2c_2 e^{-3t}$$

Take twice the first equation and add it to the second equation (which makes the c_2 terms cancel):

$2x+y = 2c_1 e^{-t}$ or $\frac{2x+y}{2c_1} = e^{-t}$

So $e^{-3t} = \left(\frac{2x+y}{2c_1}\right)^3$

Put that into the second equation:

$$y = -2c_2 \left(\frac{2x+y}{2c_1}\right)^3$$

Let $\frac{1}{r} = \frac{-2c_2}{(2c_1)^3}$

c_2 and c_1 are arbitrary.

$\Rightarrow r$ is arbitrary.

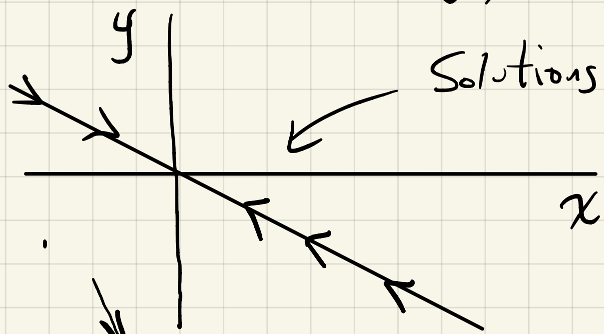
$$ry = (2x+y)^3$$

We have eliminated t and found a relationship between x and y.

Problem 1 (CONT'D)

Sketch $ry = (2x+y)^3$ for various values of r.

r=0 $(2x+y)^3 = 0 \qquad x = -\frac{1}{2}y$

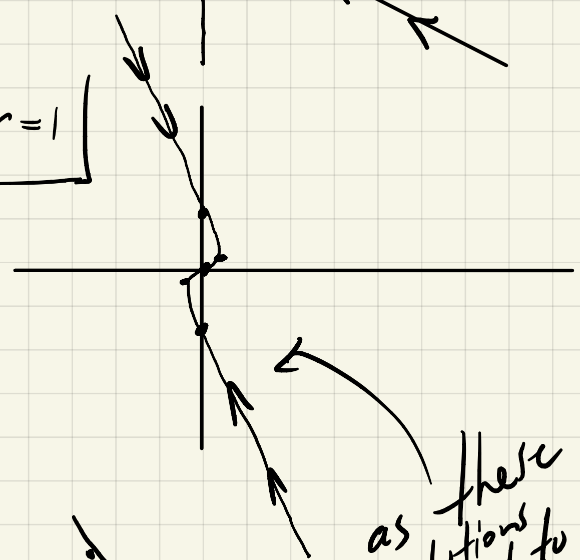


Solutions head toward origin.

$r=0 \Rightarrow c_2 = 0$

$\Rightarrow x$ and y are proportional to e^{-3t}

r=1



$y = (2x+y)^3$

some values are:

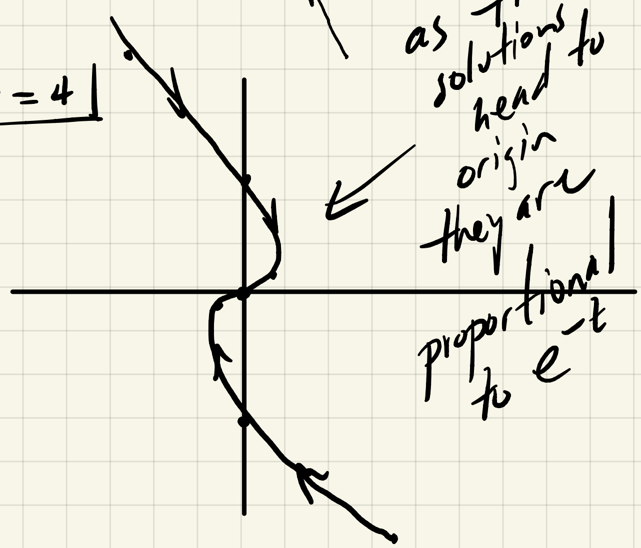
$(0,0), (0,1), (0,-1)$

$(\frac{3}{16}, \frac{1}{8}), (-\frac{3}{16}, -\frac{1}{8})$

$(-3, 8), (3, -8)$

as these solutions head to origin they are proportional to e^{-t}

r=4



$4y = (2x+y)^3$

some values are:

$(0,0), (0,2), (0,-2)$

$(\frac{3}{8}, \frac{1}{4}), (-\frac{3}{8}, -\frac{1}{4})$

$(-\frac{15}{8}, \frac{27}{4}), (\frac{15}{8}, -\frac{27}{4})$

Let us next try the nullclines + sample slopes approach.

Problem 1 (CONT'D)

The equation was $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

The x-nullcline is where $-x+y=0$. $\leftarrow y=x$

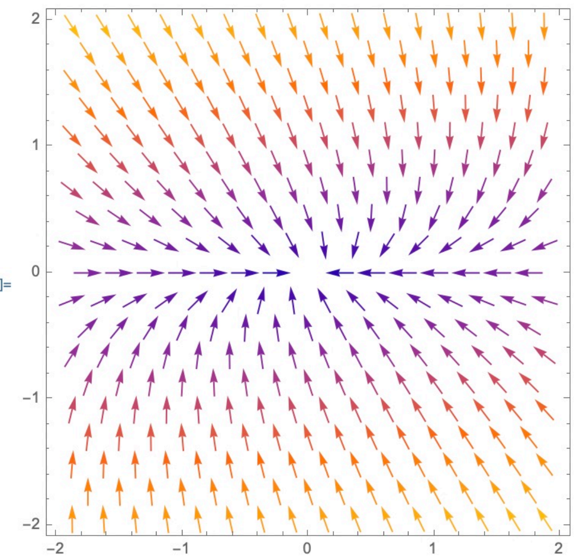
The y-nullcline is where $-3y=0$. $\leftarrow y=0$

I am going to hand the job over to Mathematica:

Phase Diagram for Logan p. 219 #3(a)

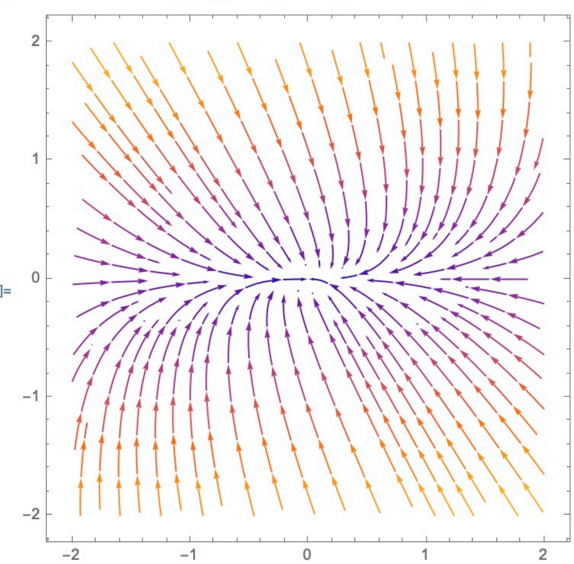
In[19]:=
```
VectorPlot[{-x + y, -3 y}, {x, -2, 2}, {y, -2, 2}]
```

Out[19]=



In[20]:=
```
StreamPlot[{-x + y, -3 y}, {x, -2, 2}, {y, -2, 2}]
```

Out[20]=



Problem 1 (CONT'D)

(b) We found: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3/5 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-4t}$.

We have

$$x = c_1 e^{t}$$
$$y = \frac{3}{5} c_1 e^{t} + c_2 e^{-4t}$$

The first equation tells us $\frac{x}{c_1} = e^{t}$. We can stick that into the second equation and get

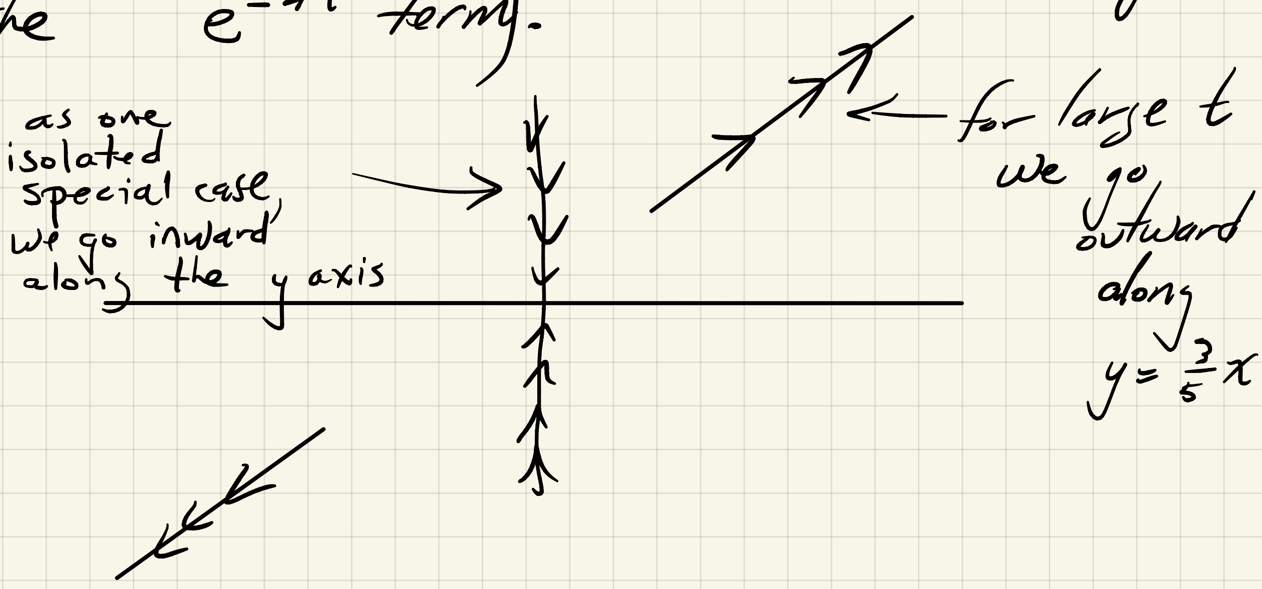
$$y = \frac{3}{5} x + c_2 \left(\frac{x}{c_1}\right)^{-4} \qquad \text{Let } c_2 c_1^4 = r$$

$$x^4 y - \frac{3}{5} x^5 = r$$

For sufficiently large x, x^4 always overwhelms r, and so for sufficiently large x, $y = \frac{3}{5} x$

So no matter what r is, eventually this approaches straight lines, expanding outward, unless $c_1 = 0$.

If $c_1 = 0$, then $x = 0$ and y contracts inward for large t (because there is only the e^{-4t} term).



Problem 1 (CONT'D) Time to switch from analytical to qualitative methods.

The equation was $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

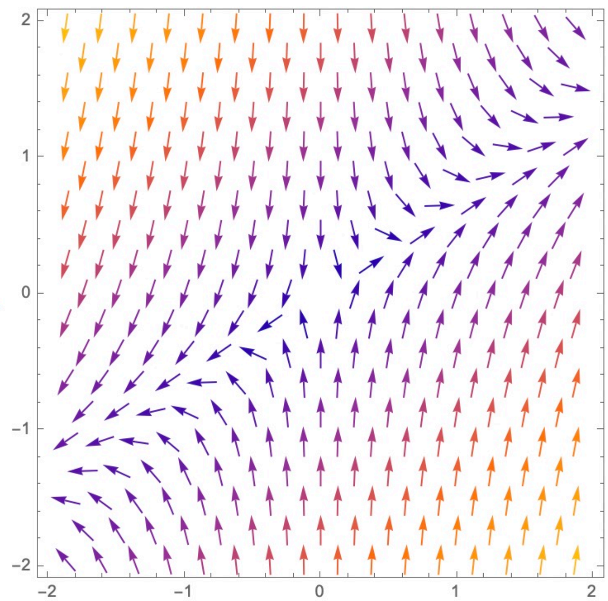
The x nullcline is $x = 0$.

The y nullcline is $3x - 4y = 0$ or $y = \frac{3}{4}x$.

Phase Diagram for Logan p. 219 #3(b)

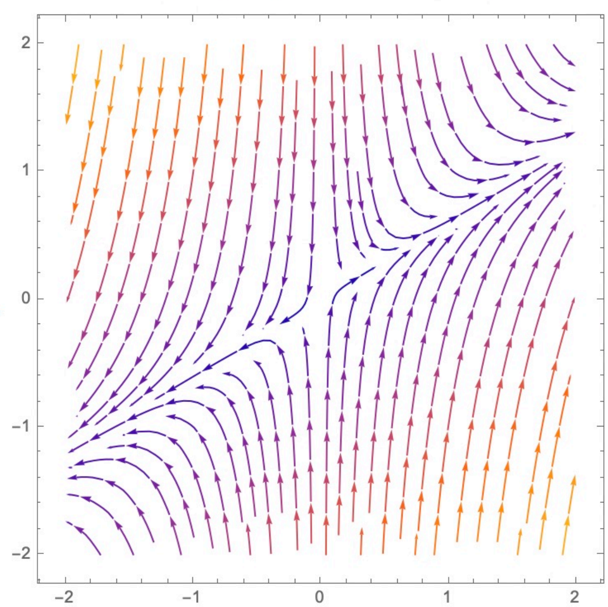
In[21]:= `VectorPlot[{x, 3 x - 4 y}, {x, -2, 2}, {y, -2, 2}]`

Out[21]=



In[22]:= `StreamPlot[{x, 3 x - 4 y}, {x, -2, 2}, {y, -2, 2}]`

Out[22]=



Problem 2 Diagrams for p. 221 #2

We found:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left[c_1 \begin{pmatrix} \cos\omega t \\ -\frac{1}{2}\cos\omega t + \underline{\omega\sin\omega t} \end{pmatrix} + c_2 \begin{pmatrix} \sin\omega t \\ \underline{-\omega\cos\omega t} + \frac{1}{2}\sin\omega t \end{pmatrix} \right] e^{\frac{3}{2}t}$$

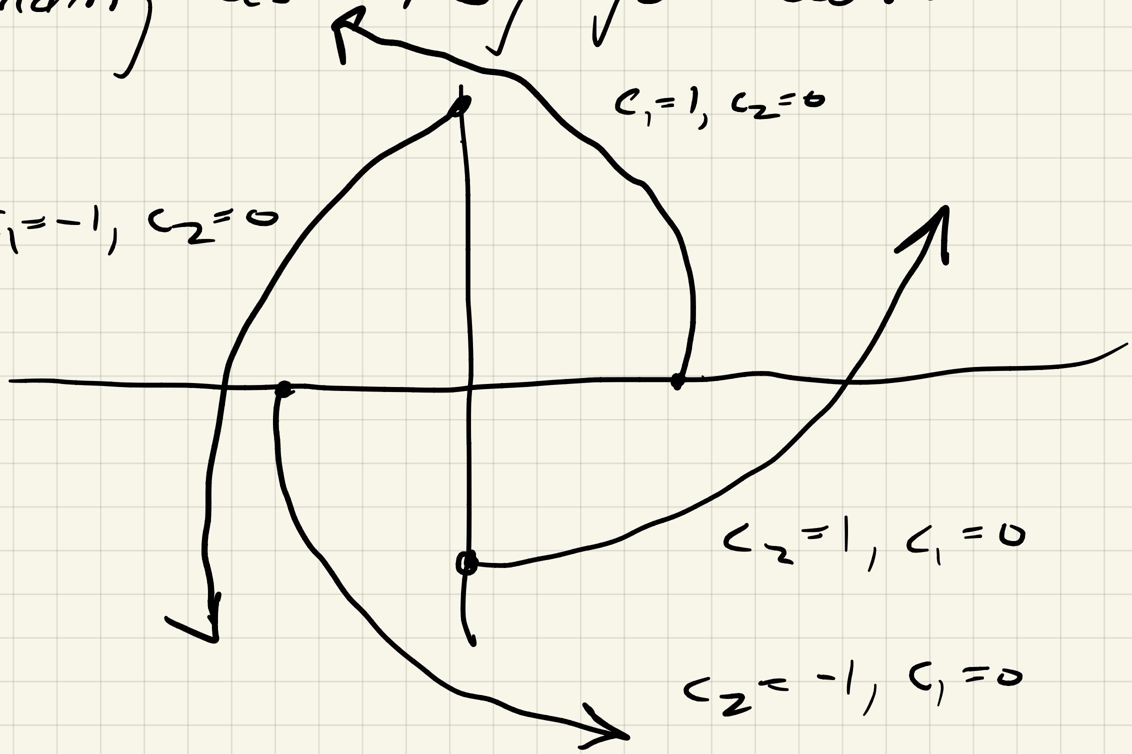
First, the $e^{\frac{3}{2}t}$ tells us solutions grow.
Second, although $\omega = \frac{\sqrt{3}}{2}$ is not a lot larger than $\frac{1}{2}$, I am going to assume the underlined terms dominate. In that case, for $c_2 = 0$, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos\omega t \\ \omega\sin\omega t \end{pmatrix} e^{\frac{3}{2}t}$$

and for $c_1 = 0$, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_2 \begin{pmatrix} \sin\omega t \\ -\omega\cos\omega t \end{pmatrix} e^{\frac{3}{2}t}$$

These are ellipses, but exponentially expanding as they go counterclockwise.

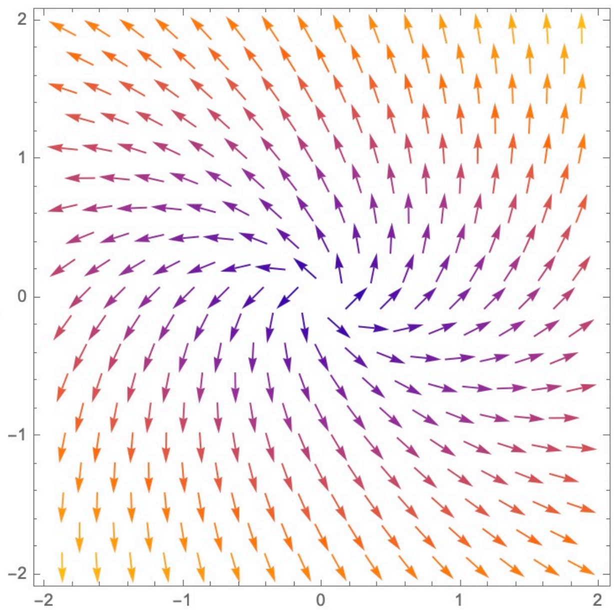


Problem 2 (CONT'D) I've gotten about as far as I am going to get with analytical methods and bad approximations. x-nullcline is $y=x$. y-nullcline is $y=-\frac{1}{2}x$.

Phase Diagram for Logan p. 221 #2

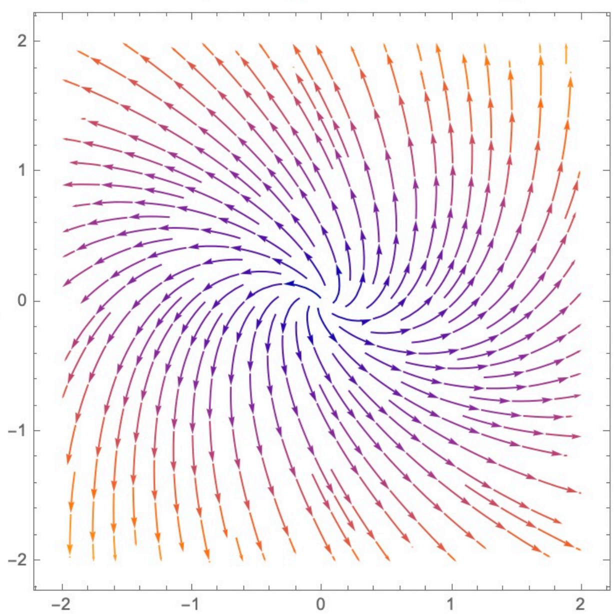
In[23]:=
```
VectorPlot[{x - y, x + 2 y}, {x, -2, 2}, {y, -2, 2}]
```

Out[23]=



In[24]:=
```
StreamPlot[{x - y, x + 2 y}, {x, -2, 2}, {y, -2, 2}]
```

Out[24]=



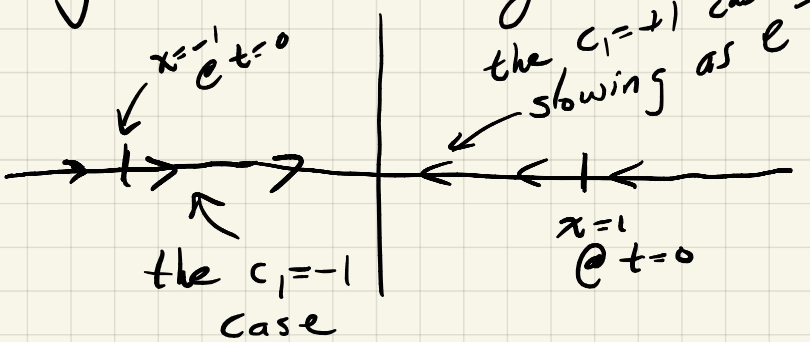
Although Mathematica completely outclasses me in production of plots, it has been very instructive to take my best shot before resorting to Mathematica.

Problem 3 Diagrams for p. 225 #1

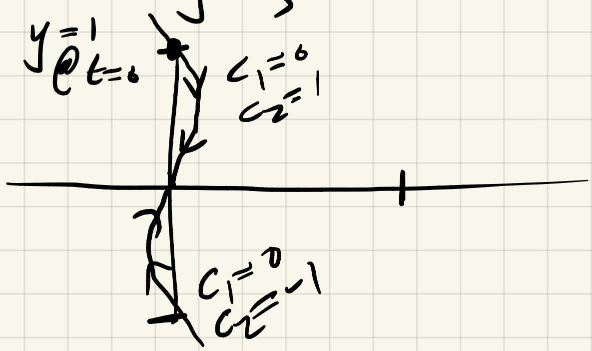
In (a) we found:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix} e^{-3t}$$

So one solution with $c_1 = 1$, $c_2 = 0$ is that x decays as e^{-3t} toward the origin while y is identically zero.



Another case to plot is $c_1 = 0$, $c_2 = 1$. The t in te^{-3t} causes x to grow but the e^{-3t} is an exponential that always wins out. The upshot is that x is t times y and y is decaying towards the origin.



That is some progress using special cases of c_1 and c_2.

Problem 3 (CONT'D)

I could also try an analytical approach.
We have $y = c_2 e^{-3t}$ regardless of c_1.
So we can solve for t.

$$t = -\frac{1}{3} \ln \frac{y}{c_2}$$

So $x = c_1 e^{-3t} + c_2 t e^{-3t}$
becomes

$$x = \frac{c_1 y}{c_2} - \frac{1}{3} y \ln \frac{y}{c_2}$$

Let's take $c_1 = 1$ and $c_2 = 1$

$$x = y - \frac{1}{3} y \ln y$$

As an approximation (which always will eventually get good as time marches on), assume that x and y are both tiny. For example, imagine that $y = e^{-9}$ (or even smaller) which happens when $t \geq 3$.
Well then $\ln y$ is a negative number, and

$-\frac{1}{3} y \ln y$ is 3 times as big as y.

So I am going to neglect y.

$$x = -\frac{1}{3} y \ln y$$

This is one solution (with $c_1 = c_2 = 1$) and it is an approximate solution only valid very near the origin.

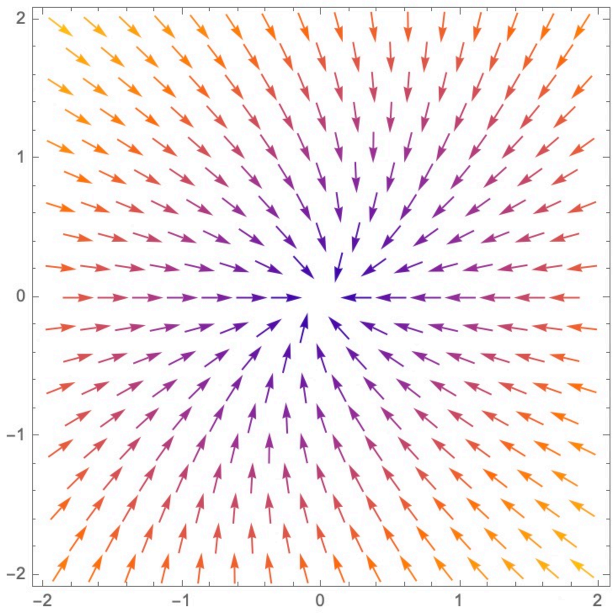
Problem 3 (CONT'D) I've gotten about as far as I am going to get with special cases and the large-t approximation.

x-nullcline is $y = 3x$. y-nullcline is $y = 0$.

Phase Diagram for Logan p. 225 #1(a)

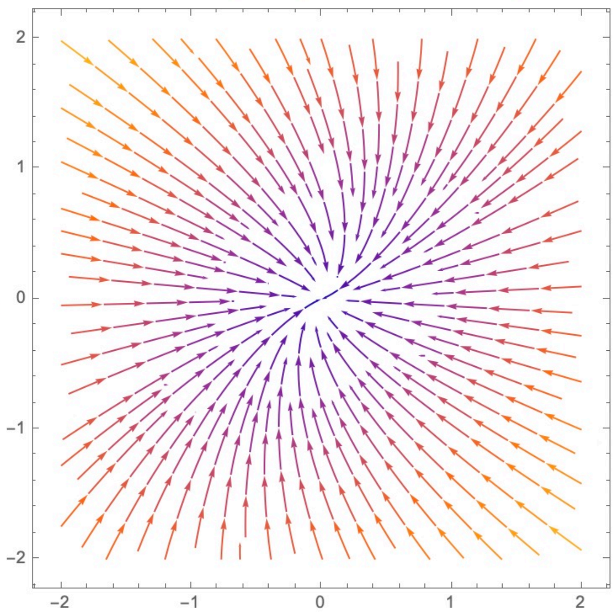
In[25]:= `VectorPlot[{-3 x + y, -3 y}, {x, -2, 2}, {y, -2, 2}]`

Out[25]=



In[27]:= `StreamPlot[{-3 x + y, -3 y}, {x, -2, 2}, {y, -2, 2}]`

Out[27]=



Problem 3 (cont'd)

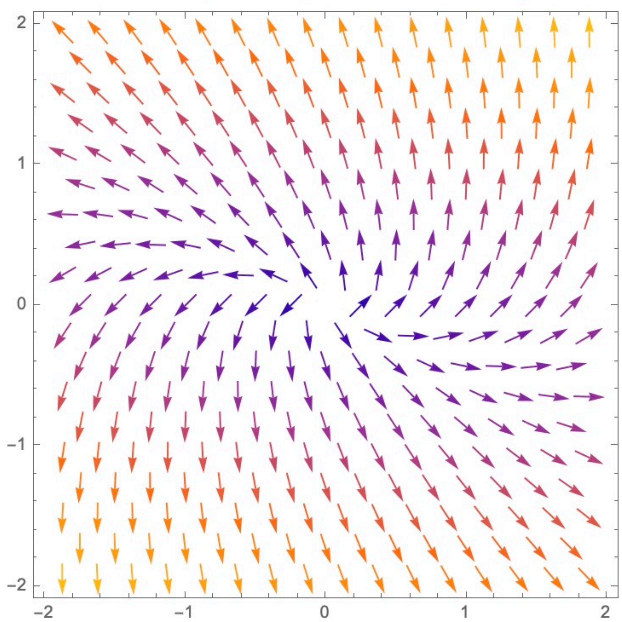
In (b) we found: $\begin{pmatrix}x\\y\end{pmatrix} = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}t\\-1-t\end{pmatrix}e^{2t}$

Since this is very similar to (a), we could play the same games, but let us go straight to the graphical method. x-nullcline is $y=x$. y-nullcline is $y=-\frac{1}{3}x$.

Phase Diagram for Logan p. 225 #1(b)

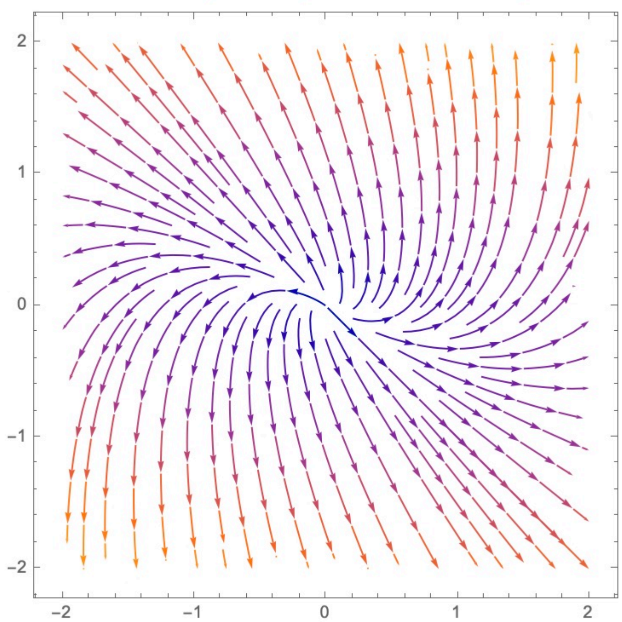
In[28]:=
```
VectorPlot[{x - y, x + 3 y}, {x, -2, 2}, {y, -2, 2}]
```

Out[28]=



In[29]:=
```
StreamPlot[{x - y, x + 3 y}, {x, -2, 2}, {y, -2, 2}]
```

Out[29]=



Problem 4 Logan p. 239 #13. An application!

Logan is asking us to put some numbers into his coupled chemical reactors example.

Therefore, applying conservation of mass to each reactor we have

$$VC_1' = qc_{\text{in}} - q_1C_1 + q_2C_2,$$
$$VC_2' = q_1C_1 - q_2C_2 - qC_2,$$

which is a linear, nonhomogeneous system for C_1 and C_2. If $c_{\text{in}} = 0$, then the system is homogeneous and has the form

$$C_1' = -\frac{q_1}{V}C_1 + \frac{q_2}{V}C_2, \tag{4.41}$$
$$C_2' = \frac{q_1}{V}C_1 - \left(\frac{q_2+q}{V}\right)C_2. \tag{4.42}$$

The coefficient matrix is

$$A = \begin{pmatrix} -\frac{q_1}{V} & \frac{q_2}{V} \\ \frac{q_1}{V} & -\frac{q_2+q}{V} \end{pmatrix}.$$

The trace is clearly negative and the determinant ($= q_1q/V^2$) is positive. Therefore the origin is asymptotically stable. Therefore the concentrations eventually go to zero. $\square$

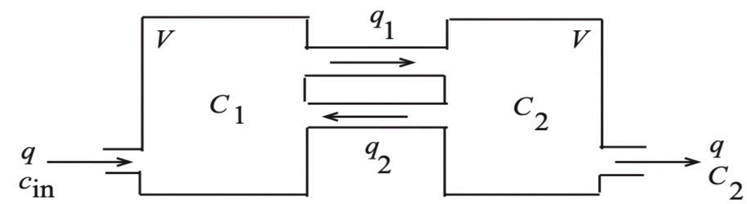


Figure 4.21 Coupled chemical reactors.

The numbers are

units of these ratios are $\frac{1}{\text{min}}$

$$\frac{q_1}{V} = \frac{16}{4} = 4, \qquad \frac{q_2}{V} = \frac{1}{4}$$

C_1 is initially 0.

C_2 is initially $\frac{0.3}{4} = \frac{3}{40}$. units are ounces per liter

Problem 4 (CONT'D) To summarize what we know so far,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}' = A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} -q_1/V & q_2/V \\ q_1/V & -(q_2+q)/V \end{pmatrix}$$

Logan has not yet used conservation of volume.

It has to be that $q + q_2 = q_1$, so we have

$$A = \begin{pmatrix} -q_1/V & q_2/V \\ q_1/V & -q_1/V \end{pmatrix} = \begin{pmatrix} -4 & 1/4 \\ 4 & -4 \end{pmatrix}$$

The nullclines are x-nullcline $y = 16x$

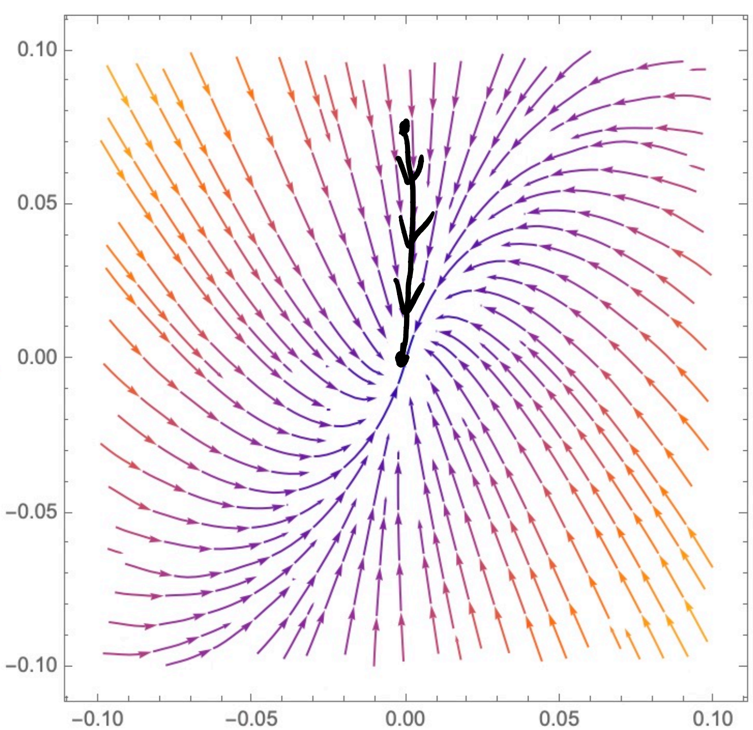
y-nullcline $y = x$

We are meant to use qualitative methods in this section. I will have Mathematica do a stream plot focused on low concentrations (since we are starting $c_1 = 0$, $c_2 = \frac{3}{40} = 0.075$)

Stream Plot for Logan p. 239 #13

```
In[30]:= StreamPlot[{-4 x + y / 4, 4 x - 4 y}, {x, -0.1, 0.1}, {y, -0.1, 0.1}]
```

Out[30]=



I have inked in the trajectory corresponding to the initial conditions.