ODE Assignment 18

To turn in on last day, June 23.

Problem 1

(i)
$$\det\begin{pmatrix} E_0-\Delta E/2-\omega & \epsilon \\ \epsilon & E_0+\Delta E/2-\omega \end{pmatrix}$$

$$= (E_0-\omega)^2-\left(\frac{\Delta E}{2}\right)^2-\epsilon^2=0$$

$$\omega_\pm = E_0 \pm \sqrt{\left(\frac{\Delta E}{2}\right)^2+\epsilon^2}$$

This $\sqrt{}$ appears all over. I am not even going to write it out.

ω_+
$$\begin{pmatrix} \cancel{E_0}-\frac{\Delta E}{2}-(\cancel{E_0}+\sqrt{}) & \epsilon \\ \epsilon & \cancel{E_0}+\frac{\Delta E}{2}-(\cancel{E_0}+\sqrt{}) \end{pmatrix}\begin{pmatrix}\psi_{10}\\ \psi_{20}\end{pmatrix}=0$$

Choose $\psi_{20}=1$ $\quad \psi_{10}=\dfrac{-\frac{\Delta E}{2}+\sqrt{}}{\epsilon}$, by the lower equation

ω_-
$$\begin{pmatrix} \cancel{E_0}-\frac{\Delta E}{2}-(\cancel{E_0}-\sqrt{}) & \epsilon \\ \epsilon & \cancel{E_0}+\frac{\Delta E}{2}-(\cancel{E_0}-\sqrt{}) \end{pmatrix}\begin{pmatrix}\psi_{10}\\ \psi_{20}\end{pmatrix}=0$$

choose $\psi_{10}=1$ $\quad \psi_{20}=\dfrac{\frac{\Delta E}{2}-\sqrt{}}{\epsilon}$, by the upper equation

(ii)
$$\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} = c_+\begin{pmatrix}\dfrac{-\frac{\Delta E}{2}+\sqrt{}}{\epsilon}\\ 1\end{pmatrix}e^{-i(E_0+\sqrt{})t}$$
$$+\,c_-\begin{pmatrix}1\\ \dfrac{\frac{\Delta E}{2}-\sqrt{}}{\epsilon}\end{pmatrix}e^{-i(E_0-\sqrt{})t}$$

With $E_0=0$:

$$\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} = c_+\begin{pmatrix}a\\ 1\end{pmatrix}e^{-i\sqrt{}\,t}+c_-\begin{pmatrix}1\\ -a\end{pmatrix}e^{i\sqrt{}\,t}$$

where $a=\dfrac{-\frac{\Delta E}{2}+\sqrt{}}{\epsilon}$

Problem 1 (CONT'D)

(iii) Particular solution with $\psi(0) = \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = c_+ \begin{pmatrix} a \\ 1 \end{pmatrix} + c_- \begin{pmatrix} 1 \\ -a \end{pmatrix} \quad \text{where} \quad a = \frac{-\frac{\Delta E}{2} + \sqrt{\ }}{\epsilon}$$

So $\quad c_+ a + c_- = 1 \quad$ and $\quad c_+ - a c_- = 0$

Or $\quad c_+ a^2 + c_- a = a \quad$ and $\quad a c_+ - a^2 c_- = 0$

add $\qquad\qquad$ subtract

$c_+(1+a^2) = a \qquad\qquad c_-(1+a^2) = 1$

$$c_+ = \frac{a}{1+a^2} \qquad c_- = \frac{1}{1+a^2}$$

We have found

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \frac{a}{1+a^2} \begin{pmatrix} a \\ 1 \end{pmatrix} e^{-i\sqrt{\ }\,t} + \frac{1}{1+a^2} \begin{pmatrix} 1 \\ -a \end{pmatrix} e^{i\sqrt{\ }\,t}$$

where $\quad a = \dfrac{-\frac{\Delta E}{2} + \sqrt{\ }}{\epsilon}$

and $\quad \sqrt{\ } = \sqrt{\left(\frac{\Delta E}{2}\right)^2 + \epsilon^2}$

Problem 1 (CONT'D)

(iv) We are asked to compute $|\psi_1(t)|^2$ where

$$\psi_1(t) = \frac{a^2 e^{-i\sqrt{\ }t} + e^{i\sqrt{\ }t}}{1+a^2}$$

$$\psi_2(t) = a\,\frac{e^{-i\sqrt{\ }t} - e^{i\sqrt{\ }t}}{1+a^2}$$

Everything has $\frac{1}{1+a^2}$ in the denominator. I'm not going to carry that around. Dealing with just numerators...

$$|\psi_1(t)|^2 = \left((1+a^2)\cos\sqrt{\ }t\right)^2 + \left((1-a^2)\sin\sqrt{\ }t\right)^2$$

$$|\psi_2(t)|^2 = 4a^2\sin^2\sqrt{\ }t$$

a is a number between 0 and 1

$a^2\sin^2\sqrt{\ }t$ is also a number between 0 and 1

$$|\psi_1(t)|^2 + |\psi_2(t)|^2 = (1+a^2)^2\cos^2\sqrt{\ }t + (1-a^2)^2\sin^2\sqrt{\ }t + 4a^2\sin^2\sqrt{\ }t$$

$$= (1+a^2)^2\cos^2\sqrt{\ }t + (1+a^2)^2\sin^2\sqrt{\ }t$$

$$= (1+a^2)^2$$

Now put back in that $\frac{1}{(1+a^2)^2}$ that we were not carrying around

$$|\psi_1(t)|^2 + |\psi_2(t)|^2 = 1$$

Or $|\psi_1(t)|^2 = 1 - |\psi_2(t)|^2$ where $|\psi_2(t)|^2 = \frac{4a^2\sin^2\sqrt{\ }t}{(1+a^2)^2}$

Problem 2

Eigenvalue equation:

$$\det \begin{pmatrix} E_0-\omega & \epsilon & \epsilon \\ \epsilon & E_1-\omega & 0 \\ \epsilon & 0 & E_1-\omega \end{pmatrix} =$$

$$(E_0-\omega)(E_1-\omega)^2 - 2\epsilon^2(E_1-\omega) = 0$$

Thankfully there is a common factor of $E_1-\omega$. So we know one root and we don't have to solve a cubic.

What remains is a quadratic

$$(E_0-\omega)(E_1-\omega) - 2\epsilon^2 = 0$$

The directions said we could approximate
$\epsilon \ll E_1 - E_0$.

Well if E_1 and E_0 are significantly different it cannot be the case that ω is both very near E_0 and very near E_1. It must be very near one or the other if the product is to be tiny.

$$(E_0-\omega)(E_1-\omega) = 2\epsilon^2$$

something tiny

If ω is near E_1 then $\quad E_1-\omega \approx \dfrac{2\epsilon^2}{E_0-E_1}$

If ω is near E_0 then $\quad E_0-\omega \approx \dfrac{2\epsilon^2}{E_1-E_0}$

Problem 2 (CONT'D)

We have found three eigenvalues (two of them only approximately).

$$\omega = E_1$$
$$\omega = E_0 - \frac{2\epsilon^2}{E_1 - E_0}$$
$$\omega = E_1 - \frac{2\epsilon^2}{E_0 - E_1}$$

Let's see if we can find the eigenvectors:

$$\omega = E_1: \quad \begin{pmatrix} E_0 - E_1 & \epsilon & \epsilon \\ \epsilon & E_1 - E_0 & 0 \\ \epsilon & 0 & E_1 - E_0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0$$

By inspection,

$\psi_1 = 0, \quad \psi_2 = 1, \quad \psi_3 = -1$ works.

This is the antisymmetric combination of the two excited states.

That suggests that the symmetric combination of ψ_2 and ψ_3 is going to be extremely useful for the remaining two eigenvalues. I'll just call that ψ_4.

Problem 2 (CONT'D)

$$\omega = E_0 - \frac{2\epsilon^2}{E_1 - E_0}:$$

$$\begin{pmatrix} \cancel{E_0} - \cancel{E_0} + \frac{2\epsilon^2}{E_1-E_0} & \epsilon & \epsilon \\ \epsilon & E_1 - E_0 + \frac{2\epsilon^2}{E_1-E_0} & 0 \\ \epsilon & 0 & E_1 - E_0 + \frac{2\epsilon^2}{E_1 - E_0} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_4 \\ \psi_6 \end{pmatrix} = 0$$

Choose $\psi_1 = 1$. Top row says:

$$\frac{\cancel{2}\epsilon^{\cancel{2}}}{E_1 - E_0} + \cancel{2}\cancel{\epsilon}\,\psi_4 = 0 \qquad \psi_4 = \frac{-\epsilon}{E_1 - E_0}$$

Is that consistent with the lower rows?

$$\epsilon + \left(E_1 - E_0 + \frac{2\epsilon^2}{E_1 - E_0}\right)\left(-\frac{\epsilon}{E_1 - E_0}\right) = 0\ ?$$

Yes, neglecting the ϵ^3 terms.

One eigenvector left to find.

Problem 2 (CONT'D)

$$\omega = E_1 - \frac{2\epsilon^2}{E_1 - E_0}:$$

$$\begin{pmatrix} E_0 - E_1 + \frac{2\epsilon^2}{E_0 - E_1} & \epsilon & \epsilon \\ \epsilon & \cancel{E_1 - E_1} + \frac{2\epsilon^2}{E_0 - E_1} & 0 \\ \epsilon & 0 & \cancel{E_1 - E_1} + \frac{2\epsilon^2}{E_0 - E_1} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_4 \\ \psi_6 \end{pmatrix} = 0$$

choose $\psi_4 = 1$. Middle row says:
(or bottom)

$$\cancel{\epsilon}\psi_1 + \frac{2\epsilon^{\cancel{2}}}{E_0 - E_1} = 0 \qquad \psi_1 = \frac{-2\epsilon}{E_0 - E_1}$$

Is that consistent with the top row? Tossing ϵ^3 terms we have

$$(E_0 - E_1)\frac{-2\epsilon}{E_0 - E_1} + 2\epsilon = 0\,?$$

Problem 2 (SUMMARIZED)

$$\omega_0 = E_0 - \frac{2\epsilon^2}{E_1 - E_0}$$

$$\omega_1 = E_1 - \frac{2\epsilon^2}{E_0 - E_1}$$

$$\omega_2 = E_1$$

We did an approximation $\epsilon << E_1 - E_0$

We have found three eigenvalues, which I have now given numbers. The corresponding eigenvectors are:

$$\psi_0 = \begin{pmatrix} 1 \\ -\epsilon/(E_1 - E_0) \\ -\epsilon/(E_1 - E_0) \end{pmatrix}$$

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2\epsilon/(E_1 - E_0) \\ 1 \\ 1 \end{pmatrix}$$

$$\psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Tashroom, because you have had QM, I have normalized the eigenvectors (to order ϵ). You can also check that they are orthogonal.

Problem 3

(i) We have to do the integral

$$\Phi(t)\int_0^t \Phi^{-1}(s)\vec{f}(s)\,ds \quad \text{where} \quad \Phi(t)=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

$$\Phi^{-1}(s)=\begin{pmatrix}\cos s & -\sin s\\ \sin s & \cos s\end{pmatrix}$$

First multiply out

$$\vec{f}(s)=\begin{pmatrix}0\\ \cos\omega s\end{pmatrix}$$

$$\Phi^{-1}(s)\vec{f}(s)=\begin{pmatrix}\cos s & -\sin s\\ \sin s & \cos s\end{pmatrix}\begin{pmatrix}0\\ \cos\omega s\end{pmatrix}=\begin{pmatrix}-\sin s\cdot\cos\omega s\\ \cos s\cdot\cos\omega s\end{pmatrix}$$

Next use $\sin A\cos B=\big(\sin(A+B)+\sin(A-B)\big)/2$

and $\cos A\cos B=\big(\cos(A+B)+\cos(A-B)\big)/2$

$$\Phi^{-1}(s)\vec{f}(s)=\frac{1}{2}\begin{pmatrix}-\sin(s+\omega s)-\sin(s-\omega s)\\ \cos(s+\omega s)+\cos(s-\omega s)\end{pmatrix} \qquad A=s,\ B=\omega s$$

Now it is easy to integrate

$$\int_0^t \Phi^{-1}\vec{f}(s)\,ds=\frac{1}{2}\begin{pmatrix}\frac{1}{1+\omega}\cos(s+\omega s)+\frac{1}{1-\omega}\cos(s-\omega s)\\ \frac{1}{1+\omega}\sin(s+\omega s)+\frac{1}{1-\omega}\sin(s-\omega s)\end{pmatrix}\Bigg|_0^t$$

$$=\frac{1}{2}\begin{pmatrix}\frac{1}{1+\omega}\big(\cos(t+\omega t)-1\big)+\frac{1}{1-\omega}\big(\cos(t-\omega t)-1\big)\\ \frac{1}{1+\omega}\sin(t+\omega t)+\frac{1}{1-\omega}\sin(t-\omega t)\end{pmatrix}$$

We still have to multiply this whole mess by $\Phi(t)$.

Problem 3 (CONT'D)

(ii) We focus on the terms that are large for ω near 1.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx \frac{1}{2}\frac{1}{1-\omega}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos(t-\omega t)-1 \\ \sin(t-\omega t) \end{pmatrix}$$

$$= \frac{1}{2}\frac{1}{1-\omega}\begin{pmatrix} \cos t\cos(t-\omega t) - \cos t + \sin t \sin(t-\omega t) \\ -\sin t \cos(t-\omega t) + \sin t + \cos t \sin(t-\omega t) \end{pmatrix}$$

$$= \frac{1}{2}\frac{1}{1-\omega}\begin{pmatrix} \cos \omega t - \underline{\cos t} \\ -\sin \omega t \ \underline{+\sin t} \end{pmatrix}$$

I used

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

with $A = t$

$B = t - \omega t$

$A - B = \omega t$

I am kind of surprised by the underlined terms. I would not expect anything that is not at the driving frequency to blow up.