ODE Midterm Solution

1. Separation of Variables – Logan, p.27 #12

(a) Solve the initial value problem

$$y' = \frac{2ty}{1+t^2} \qquad y(0) = y_0$$

$$\frac{y'}{y^2} = \frac{2t}{1+t^2} \quad \text{or} \quad \left(-\frac{1}{y}\right)' = \frac{2t}{1+t^2}$$

$$-\frac{1}{y} = \int \frac{2t\,dt}{1+t^2} = \ln(1+t^2) + c$$

$$y = -\frac{1}{\ln(1+t^2)+c}$$

Plug in $t=0$ to see that

$$c = -\frac{1}{y_0}$$

$$y = \frac{1}{\ln\frac{1}{1+t^2} + \frac{1}{y_0}}$$

Problem 1 (CONT'D)

Good to double-check before proceeding

$$y = \frac{1}{\ln \frac{1}{1+t^2} + \frac{1}{y_0}}$$

$$y' = \frac{-1}{\left(\ln \frac{1}{1+t^2} + \frac{1}{y_0}\right)^2} (1+t^2) \frac{-1}{(1+t^2)^2} 2t$$

$$= y^2 \frac{2t}{1+t^2} \quad \checkmark\checkmark$$

Part (b) — Interval of existence

$y_0 < 0$ is easy, because $\ln \frac{1}{1+t^2}$ is always ≤ 0, so $\ln \frac{1}{1+t^2} + \frac{1}{y_0} < 0$ and solution is good everywhere.

$y_0 = 0$ is easy, but our answer doesn't make it obvious. The sol'n is $y(t) \equiv 0$ and it is good everywhere.

Finally $y_0 > 0$. We have trouble when $\ln \frac{1}{1+t^2} + \frac{1}{y_0} = 0$. This happens when $1+t^2 = e^{-1/y_0}$. I.e., $-\sqrt{e^{-1/y_0} - 1} < t < \sqrt{e^{-1/y_0} - 1}$

the interval of existence

2. Integrating factors – Logan p. 42 #10

(a) Find the general solution to

$$x' - px = q(t)$$

p ← a constant

Integrating factor is e^{-pt}

$$e^{-pt}x' - pxe^{-pt} = (xe^{-pt})' \checkmark$$

So
$$xe^{-pt} = \int_0^t q(s)\,ds + c$$

(b) Find c such that $x(t_0) = x_0$

$$x_0 e^{-pt_0} = \int_0^{t_0} q(s)\,ds + c$$

So $c = x_0 e^{-pt_0} - \int_0^{t_0} q(s)\,ds$

or
$$x(t) = e^{pt}\left[\int_{t_0}^{t} q(s)\,ds + x_0 e^{-pt_0}\right]$$

3. Resonance — Logan p. 114 #2

We have an LC circuit driven at resonance

$$LQ'' + \frac{1}{C}Q = V_0 \sin\frac{t}{\sqrt{LC}}$$

or
$$Q'' + \omega_0^2 Q = \frac{V_0}{L}\sin\omega_0 t$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$. Logan chooses $\frac{V_0}{L} = 1$. I am not sure what that means. $\frac{V_0}{L}$ is not a dimensionless quantity. But we will use his form:

$$Q'' + \omega_0^2 Q = \sin\omega_0 t$$

He says the particular solution is

$$Q_p(t) = t(A\sin\omega_0 t + B\cos\omega_0 t)$$

$$Q_p'(t) = A\sin\omega_0 t + B\cos\omega_0 t + t(A\omega_0\cos\omega_0 t - B\omega_0\sin\omega_0 t)$$

$$Q_p''(t) = 2(A\omega_0\cos\omega_0 t - B\omega_0\sin\omega_0 t) + t(-A\omega_0^2\sin\omega_0 t - B\omega_0^2\cos\omega_0 t)$$

Problem 3 (CONT'D)

Put the particular solution in:

$$2(A\omega_0 \cos\omega_0 t - B\omega_0 \sin\omega_0 t)$$
$$+ t\left(\cancel{-A\omega_0^2 \sin\omega_0 t} - \cancel{B\omega_0^2 \cos\omega_0 t}\right)$$
$$+ \omega_0^2 t \left(\cancel{A\sin\omega_0 t + B\cos\omega_0 t}\right) = \sin\omega_0 t$$

So $A = 0$ and $-2\omega_0 B = 1$

So $Q_p(t) = -\dfrac{1}{2\omega_0} t\cos\omega_0 t$

Now we add the general solution of the homogeneous equation:

$$Q(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t - \frac{1}{2\omega_0} t\cos\omega_0 t$$

$Q(0) = 0$ implies $c_1 = 0$.

$Q'(0) = c_2\omega_0 - \dfrac{1}{2\omega_0}$ implies $c_2 = \dfrac{1}{2\omega_0^2}$

So our final answer is

$$Q(t) = \frac{1}{2\omega_0^2} \sin\omega_0 t - \frac{1}{2\omega_0} t\cos\omega_0 t$$
$$= \frac{1}{2\omega_0^2}\left(\sin\omega_0 t - \omega_0 t \cos\omega_0 t\right)$$

Problem 4 — Cauchy–Euler Equations — similar to Logan Example 2.30 on p. 119

$$x'' - \frac{2}{t}x' + \frac{2}{t^2}x = 0$$

Try $x = t^m$. Get indicial equation:

$$m(m-1)t^{m-2} - \frac{2m}{t}t^{m-1} + \frac{2}{t^2}t^m = 0$$

$$m^2 - m - 2m + 2 = 0 \quad \text{or} \quad m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0 \qquad m = 1 \text{ or } m = 2$$

So general solution is:

$$x(t) = c_1 t + c_2 t^2$$

Problem 5 — Reduction of Order — Logan p. 125 #2

$$x_1(t) = t,\ x_1'(t) = 1,\ x_1''(t) = 0$$

So $x_1'' - t x_1' + x_1 = 0 - t \cdot 1 + t = 0$ ✓

Try $x_2(t) = t v(t)$

$x_2'' = 2v' + tv''$; $x_2' = v + tv'$

$$x_2'' - t x_2' + x_2 = 2v' + tv'' - t(\cancel{v} + tv') + \cancel{tv}$$
$$= tv'' + 2v' - t^2 v'$$

Problem 5 (CONT'D)

Let $w = v'$ $\qquad tw' + 2w - t^2 w = 0$

$$w' + \frac{2}{t}w - tw = 0$$

$$w' + \frac{2}{t}w - tw = 0$$

can be solved with the integrating factor method.

$$\left(we^{+2\ln t - t^2/2}\right)' = 0$$

$$w = e^{-2\ln t + t^2/2}$$