ODE Final Solution

Problem 1

a consequence of $f(t+p)=f(t)$ —
a mathematician might want a proof by induction of this initial claim

(i) $f(t)$ is periodic with period p.

This means $f(t+np)=f(t)$ for any integer n.

$$F(s) \equiv \int_0^\infty f(t)e^{-st}\,dt$$

$$= \underbrace{\sum_{n=0}^\infty \int_{np}^{(n+1)p} f(t)e^{-st}\,dt}$$

what I have done here is broken the integral up into an infinite sum of integrals, each covering the range from np to $(n+1)p$

$$= \sum_{n=0}^\infty \underbrace{\int_0^p f(r+np)e^{-s(r+np)}\,dr}$$

Change of variables $t=r+np$

$$= \sum_{n=0}^\infty \underbrace{e^{-snp} \int_0^p f(r)e^{-sr}\,dr}$$

Here I have pulled out e^{-snp} because it is independent of r and also I used $f(r+np)=f(r)$

$$= \underbrace{\frac{1}{1-e^{-sp}}} \int_0^p f(r)e^{-sr}\,dr$$

Here I have used $e^{-snp}=\left(e^{-sp}\right)^n = z^n$

and $\sum_{n=0}^\infty z^n = \frac{1}{1-z}$

with $z=e^{-sp}$

(ii) Our function has period T. So what we derived in (i) says

$$F(s) = \frac{1}{1-e^{-sT}} \int_0^T f(r)\, e^{-sr}\, dr$$

In this case, $f(r) = \frac{1}{T} r$ ← including the $\frac{1}{T}$ is what I meant by "get the slope right."

$$= \frac{1}{1-e^{-sT}} \frac{1}{T} \int_0^T r e^{-sr}\, dr$$

$$= \frac{1}{1-e^{-sT}} \frac{1}{T} \left(-\frac{d}{ds}\right) \int_0^T e^{-sr}\, dr$$

$$= \frac{1}{1-e^{-sT}} \frac{1}{T} \left(+\frac{d}{ds}\right) \left(+\frac{1}{s} e^{-sr}\right)\Big|_0^T$$

$$= \frac{1}{1-e^{-sT}} \frac{1}{T} \frac{d}{ds} \left(\frac{1}{s}\left(e^{-sT}-1\right)\right)$$

$$= \frac{1}{1-e^{-sT}} \frac{1}{T} \left[\left(-\frac{1}{s^2}\right)\left(e^{-sT}-1\right) - \frac{T}{s} e^{-sT}\right]$$

Perhaps we can tidy this up a little by multiplying numerator and denominator through by e^{sT}....

$$F(s) = \frac{1}{1-e^{sT}} \frac{1}{Ts^2} \left(1-e^{sT} + Ts\right)$$

I also pulled out $\frac{1}{s^2}$, and absorbed a minus sign by changing $\frac{1}{e^{sT}-1}$ to $\frac{1}{1-e^{sT}}$

Problem 2

(i) Solve $x' + x = H(t-1) - H(t-2)$ with $x(0) = 1$

By Equation (3.6), the Laplace transform of x' is $sX(s) - x(0)$. So the LHS is

$sX(s) - 1 + X(s)$

The RHS is $(e^{-s} - e^{-2s})/s$

So $X(s) = \left(1 + \frac{e^{-s} - e^{-2s}}{s}\right) \Big/ (1+s)$

Let's deal with the three terms, one at a time.

$\mathcal{L}^{-1}\left[\frac{1}{1+s}\right](t) = e^{-t}$

The second term is $H(t-1)f(t-1)$ where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\frac{1}{1+s}\right](t)$$
$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{1+s}\right](t)$$
$$= 1 - e^{-t}$$

The third term is $H(t-2)f(t-2)$. In summary,

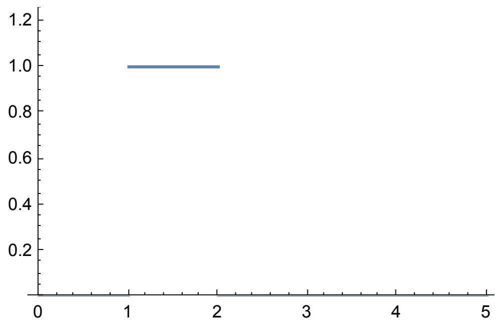
$$x(t) = e^{-t} + H(t-1)\left[1 - e^{-(t-1)}\right] - H(t-2)\left[1 - e^{-(t-2)}\right]$$

Problem 2 (ii) Graphs

You were only expected to do sketches on these problems, but since I have Mathematica handy, I am giving the solution with high-quality graphs.

In[25]:=
```
Plot[HeavisideTheta[t - 1] - HeavisideTheta[t - 2],
 {t, 0, 5}, PlotRange → {Automatic, {0, 1.25}}]
```

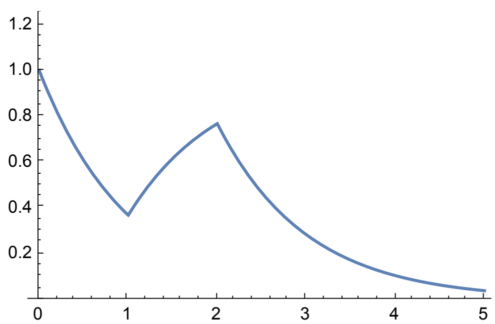
Out[25]=



In[26]:=
```
xi[t_] := Exp[-t] + HeavisideTheta[t - 1] (1 - Exp[-(t - 1)]) -
   HeavisideTheta[t - 2] (1 - Exp[-(t - 2)])
```

In[27]:=
```
Plot[xi[t], {t, 0, 5}, PlotRange → {Automatic, {0, 1.25}}]
```

Out[27]=

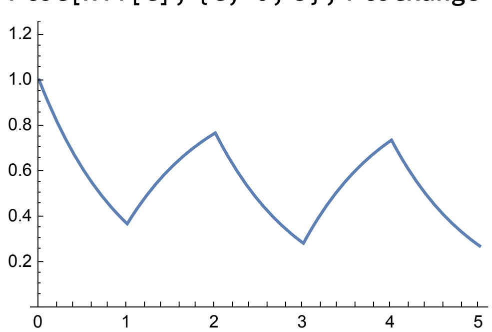


Problem 2 (iii) Graphs

In[28]:=
```
xii[t_] := xi[t] + HeavisideTheta[t - 3] (1 - Exp[-(t - 3)]) -
   HeavisideTheta[t - 4] (1 - Exp[-(t - 4)])
```

In[29]:=
```
Plot[xii[t], {t, 0, 5}, PlotRange → {Automatic, {0, 1.25}}]
```

Out[29]=



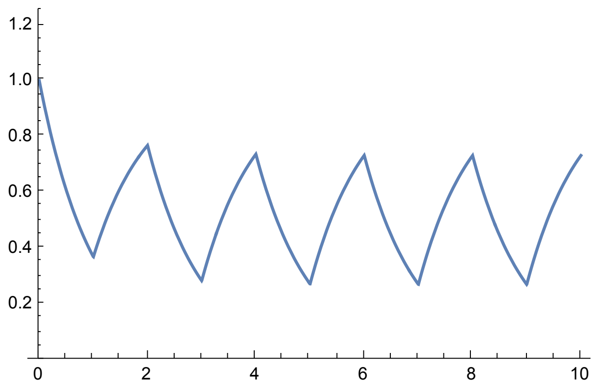
Comment — Sawtooth Waves and Triangle Waves

In Problem 1, you examined sawtooth waves. These have many applications, including pulse width modulation (PWM), which is how a modern inverter makes AC power from DC power (such as that which might come from a battery or solar cell). This is accomplished with high-frequency switching power supplies using PWM.

The Problem 2 graph is starting to settle down to something that is close to a triangle wave. It has mean 0.5 and the ups and downs aren't perfectly straight, but it is nonetheless pretty close. The triangle wave is a close relative of the sawtooth wave that you studied in the first problem. Below is a plot with more cycles.

```
In[•]:= xComment[t_] := xi[t] +
  HeavisideTheta[t - 3] (1 - Exp[-(t - 3)]) - HeavisideTheta[t - 4] (1 - Exp[-(t - 4)]) +
  HeavisideTheta[t - 5] (1 - Exp[-(t - 5)]) - HeavisideTheta[t - 6] (1 - Exp[-(t - 6)]) +
  HeavisideTheta[t - 7] (1 - Exp[-(t - 7)]) - HeavisideTheta[t - 8] (1 - Exp[-(t - 8)]) +
  HeavisideTheta[t - 9] (1 - Exp[-(t - 9)])
```

```
In[•]:= Plot[xComment[t], {t, 0, 10}, PlotRange → {Automatic, {0, 1.25}}]
```

Out[•]=


The radio and computer industries created a lot of demand for waveform generation. As an important example, waveform generation is what got Hewlett-Packard its start before WWII. In the 1960s, the specialized test equipment came down in price and percolated into the hands of musicians who were often professional or amateur electrical engineers.

At first the sound of the new waveforms was just a novelty, but in the hands of geeks who also possessed musical talent, a new form of music called electronic music evolved. The basic waveforms (triangle waves, square waves, sawtooth waves) sound lousy until they are patiently mixed. Synthesizer companies got popular in the 1970s that made adjusting the waveforms easier. As examples of the genre, try:

* *On The Run, https://youtu.be/POKIpg6NGzQ,* by Pink Floyd from *The Dark Side of the Moon*
* The soundtrack to *Blade Runner.*

The Pink Floyd YouTube link is clickable.

Problem 3

(i) $\cosh^2 x - \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

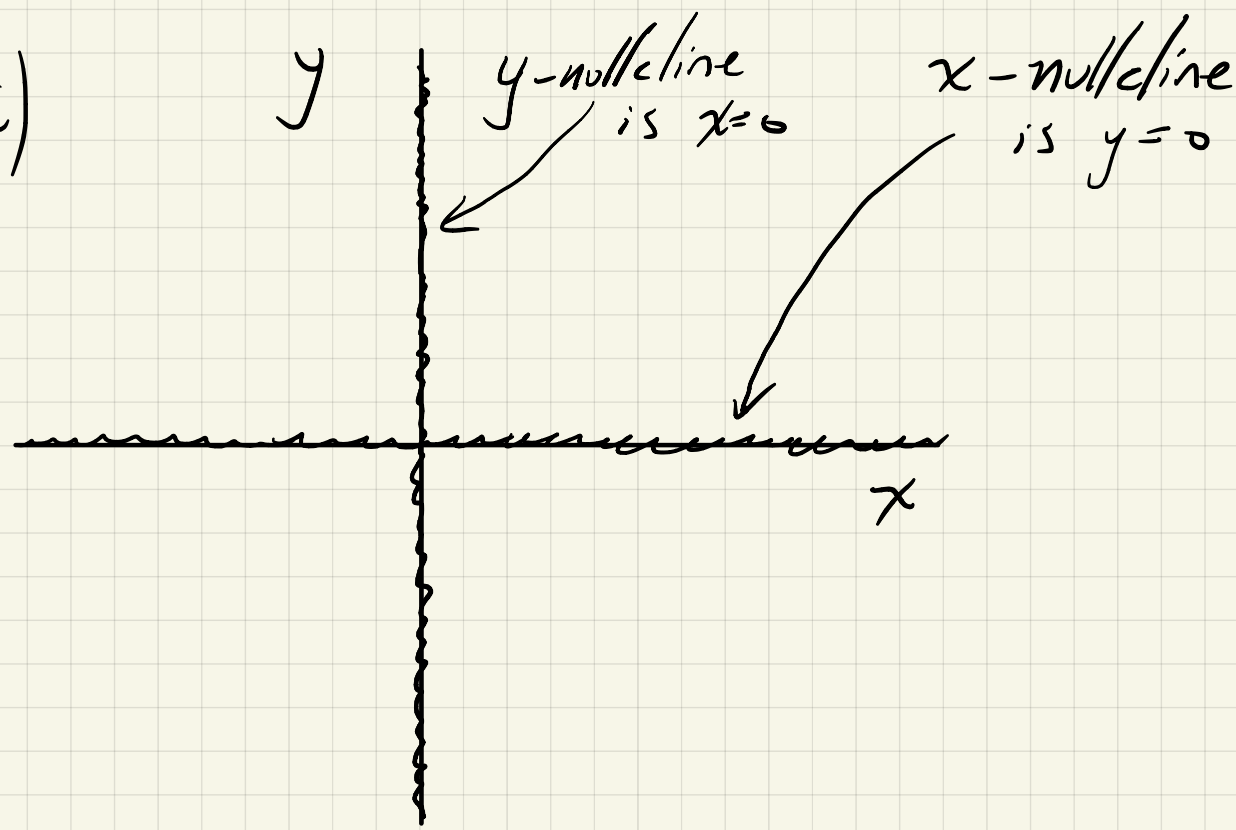
$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}}{4}$$

$$= \frac{4}{4} = 1$$

(ii) $\frac{x}{3} = \cosh t \qquad y = \sinh t$

By part (i) $\qquad \left(\frac{x}{3}\right)^2 - y^2 = 1$

(iii)

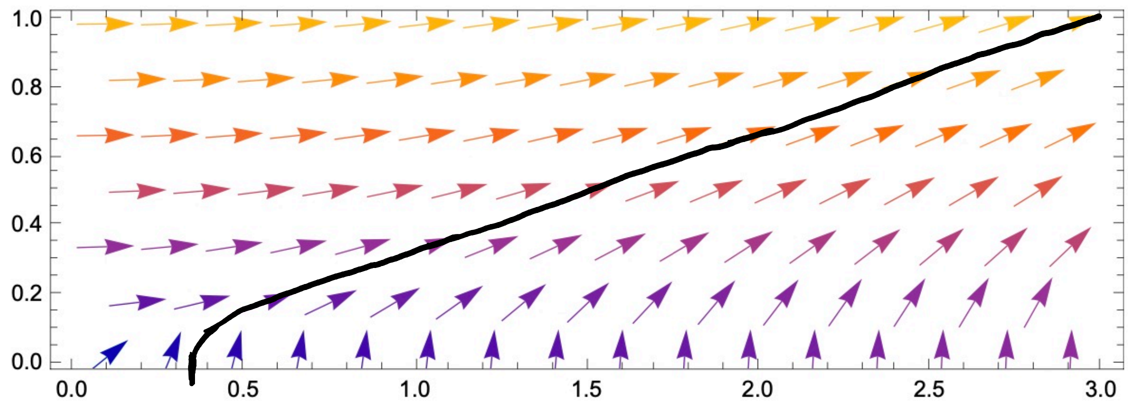


Problem 3 (iv) Table and Graph

x	y	x'	y'
1 / 3	1 / 9	1 / 3	1 / 9
1 / 3	2 / 9	2 / 3	1 / 9
1 / 3	1 / 3	1	1 / 9
2 / 3	1 / 9	1 / 3	2 / 9
2 / 3	2 / 9	2 / 3	2 / 9
2 / 3	1 / 3	1	2 / 9
1	1 / 9	1 / 3	3 / 9
1	2 / 9	2 / 3	3 / 9
1	1 / 3	1	3 / 9

```
In[32]:= VectorPlot[{3 y, x / 3}, {x, 0, 3}, {y, 0, 1}, AspectRatio → 1 / 3]
```

Out[32]=



(v) I have inked in the trajectory.

Note that the Mathematica plot above is nice, but isn't quite what was asked for. Mathematica makes all the vectors have the same length. In other words, Mathematica's vector plot is only showing directions, not magnitudes.

Problem 4 Logan p. 238 #7

(a) $$\vec{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \vec{x}$$

The critical point has $x' = y' = 0$ (by definition), and this occurs at the origin.

The eigenvalue equation is

$$\det \begin{pmatrix} -1-\lambda & -1 \\ -\alpha & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1-\lambda) - (-1)(-\alpha) = 0$$

Canceling a lot of minus signs, we have

$$(\lambda+1)^2 = \alpha \quad \text{so} \quad \lambda_{\pm} = -1 \pm \sqrt{\alpha}$$

For λ_+

$$\begin{pmatrix} -\sqrt{\alpha} & -1 \\ -\alpha & -\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = 0$$

If we choose $x_+ = \sqrt{\alpha}$ then $y_+ = -1$

For λ_-

$$\begin{pmatrix} \sqrt{\alpha} & -1 \\ -\alpha & \sqrt{\alpha} \end{pmatrix} \begin{pmatrix} x_- \\ y_- \end{pmatrix} = 0$$

If we choose $x_- = \sqrt{\alpha}$ then $y_- = 1$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_+ \begin{pmatrix} \sqrt{\alpha} \\ -1 \end{pmatrix} e^{(-1+\sqrt{\alpha})t} + c_- \begin{pmatrix} \sqrt{\alpha} \\ 1 \end{pmatrix} e^{(-1-\sqrt{\alpha})t}$$

For $\alpha = \frac{1}{2}$, $\sqrt{\alpha} = \frac{\sqrt{2}}{2} < 1$ so $-1+\sqrt{\alpha}$ and $-1-\sqrt{\alpha}$ are both negative.

The origin is completely stable.

All solutions are driven to the origin.

Problem 4 (CONT'D)

(b) The only difference if $\alpha = 2$ is that the roots are $-1 \pm \sqrt{2}$, but this is a big difference. It means that there is a positive eigenvalue and a negative eigenvalue. The positive exponential always wins.

$$\vec{x}(t) = c_+ \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{(-1+\sqrt{2})t} + c_- \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{(-1-\sqrt{2})t}$$

<u>Except</u> along the line where $c_+ = 0$ solutions <u>always</u> blow up.

(c) Discuss $\frac{1}{2} < \alpha < 2$

The big transition point is at $\alpha = 1$.

When $\alpha = 1$ we have two solutions, one decaying and one constant.

When $\alpha < 1$ we have the situation in (a).
When $\alpha > 1$ we have the situation in (b).
Looking a little more at the $\alpha = 1$ case:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$$

The critical <u>line</u> is $x = -y$. Solutions are driven to this line and rest anywhere along it.

Problem 5 Logan p. 245 #5

(i) $\vec{x}' = A\vec{x} + \vec{f}$ where $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

and $\vec{f} = \begin{pmatrix} 0 \\ \cos\omega t \end{pmatrix}$ $\omega \neq \pm 1$

To get the fundamental matrix $\underline{\Phi}$ we have to find two independent solutions.

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$\lambda = i$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0 \Rightarrow -ix_0 + y_0 = 0$$

So if $x_0 = 1$, then $y_0 = i$

So one solution of the homogeneous equation is

$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}$ and so is its complex conjugate $\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}$

And so is their sum divided by 2

$$\begin{pmatrix} \dfrac{e^{it} + e^{-it}}{2} \\ \dfrac{ie^{it} - ie^{-it}}{2} \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

And their difference divided by $2i$

$$\begin{pmatrix} \dfrac{e^{it} - e^{-it}}{2i} \\ \dfrac{ie^{it} + ie^{-it}}{2i} \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

So the complete set of real solutions to the homogeneous equation is

$$\vec{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Problem 5 (CONT'D)

(ii) Well, we have our fundamental matrix:

$$\Phi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad \Phi^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

We have to use equation (4.45) to get the particular solution of the nonhomogeneous equation.

$$\vec{x}(t) = \Phi \vec{k} + \Phi \int_0^t \Phi^{-1} \vec{f}(s)\, ds$$

Since Logan didn't specify initial conditions $\vec{k} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$ is anything. Whatever the initial condition, to get a particular solution requires us to do the integral

$$\vec{x}(t) = \Phi \int_0^t \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ \cos \omega s \end{pmatrix} ds$$

$$= \Phi \int_0^t \begin{pmatrix} -\sin s \cdot \cos \omega s \\ \cos s \cdot \cos \omega s \end{pmatrix} ds$$

Since it is a bit of a mess to push this through and you did not (either of you) have sufficient time, we will tackle this again as the last problem on Assignment 18.

Even though it is a bit of a mess, it is an important resonance problem. The system has natural frequency ω_0 (which Logan has chosen the constants such that $\omega_0 = 1$). The system is being driven at some other frequency $\omega \neq \omega_0$. You will see that there are $\frac{1}{\omega - \omega_0}$ terms in the solution. The case $\omega = \omega_0$ must be handled separately.