Differential Equations Final

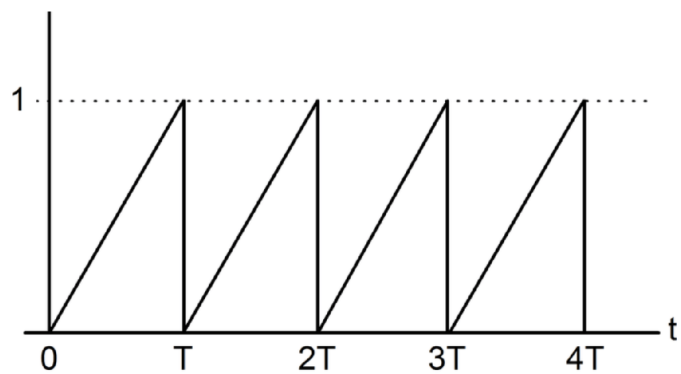
9:00-11:00am, June 21, 2022

Open-textbook (but otherwise not open-note nor other resources).

Problem 1 — Laplace Transforms — Periodic Functions

(i) Do the proof Logan requests in p. 158 #16, starting straight from the definition of the Laplace transform, Equation (3.1), and use Logan's hint.

(ii) Apply the result from (i) to find the Laplace transform of the sawtooth function of unit height and period T graphed below. (Check to get the slope right so that our answers agree.)



Problem 2 — Laplace Transforms — Heaviside Functions

(i) Solve Logan, p. 158 #14 using Laplace transforms.

(ii) Sketch $H(t-1) - H(t-2)$ and sketch your solution.

(iii) Without doing any more analytical work, sketch the solution to

$x' = -x + H(t-1) - H(t-2) + H(t-3) - H(t-4),\ \ x(0) = 1.$

Problem 3 — Linear Systems — Parametric Curves

One solution of $x' = 3y$, $y' = x/3$, with $x(0) = 3$, $y(0) = 0$, is $x = 3\cosh t$ and $y = \sinh t$. In this problem, we explore the parametric curve and the phase space.

(i) Using the definitions of $\cosh x \equiv (e^x + e^{-x})/2$, and $\sinh x \equiv (e^x - e^{-x})/2$, show that $\cosh^2 x - \sinh^2 x = 1$.

(ii) If $x = 3\cosh t$ and $y = \sinh t$, what is the relation between x and y?

(iii) Describe the x and y nullclines.

(iv) Plot (as Logan does in Figure 4.3) some vectors showing the flow in phase space. The following nine vectors in the first quadrant should be sufficient since adjacent quadrants are reflections of each other :

x	y	x '	y '
1 / 3	1 / 9		
1 / 3	2 / 9		
1 / 3	1 / 3		
2 / 3	1 / 9		
2 / 3	2 / 9		
2 / 3	1 / 3		
1	1 / 9		
1	2 / 9		
1	1 / 3		

(v) Use the information gained in parts (iii) and (iv) to sketch the trajectory beginning with $x(0) = 1/3$, $y(0) = 0$.

Problem 4 — Linear Systems — Bifurcation

Logan p. 238 #7. You can be very brief in part (c), but do discuss the value of α where the stability properties change.

Problem 5 — Nonhomogeneous Linear Systems

Logan p. 245 #5. Please split your solution into two parts:

(i) Find the complete set of real solutions to the homogeneous equation.

(ii) Find the particular solution.