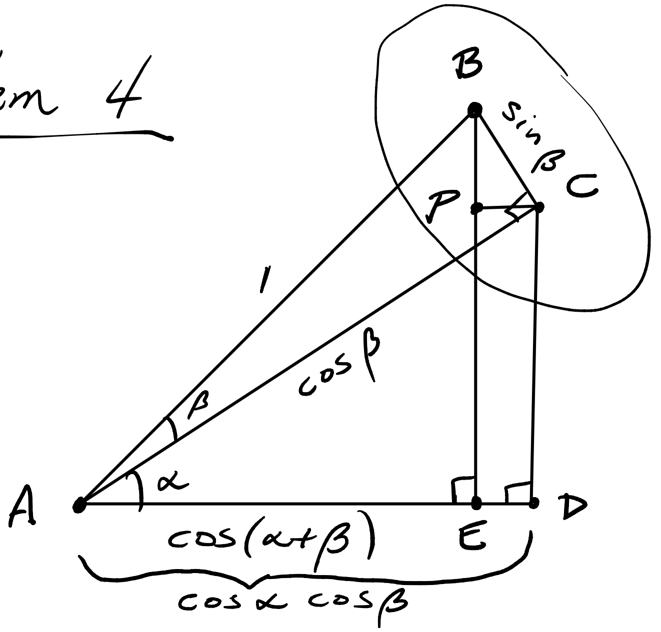
Problem Set 1

Problems 4, 5, 6, and 10 from Chapter 1 of Van Brummelen

Problem 4



SHOW:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$\overline{AB}$ has length 1. By assumption.

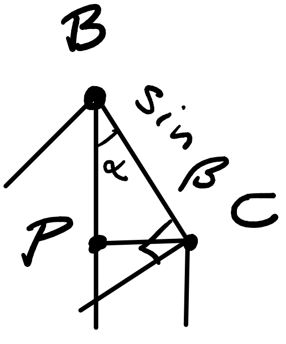
$\overline{AC}$ has length $\cos\beta$.

$\overline{AE}$ is what we are looking for! $\cos(\alpha+\beta)$.

$\overline{AD}$ has length $\cos\alpha \cos\beta$.

$\overline{BC}$ has length $\sin\beta$

Zoom in on the circled part.



The top angle of triangle BPC is α!

There are a couple of way of seeing why.

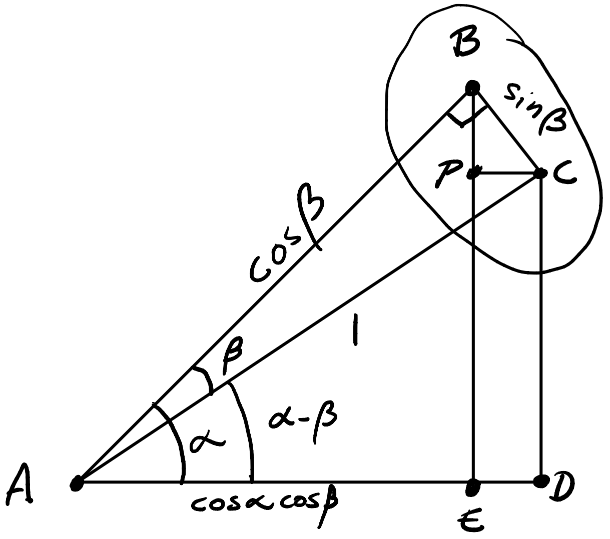
So $\overline{PC}$ has length $\sin\alpha \sin\beta$. And we are done because we <u>now</u> have two expressions for the length of $\overline{AD}$: (1) It is $\cos(\alpha+\beta) + \sin\alpha\sin\beta$. (2) It is $\cos\alpha\cos\beta$.

(Equate the two expressions and move $\sin\alpha \sin\beta$ to the RHS)

PROBLEM 4 (CONT'D)

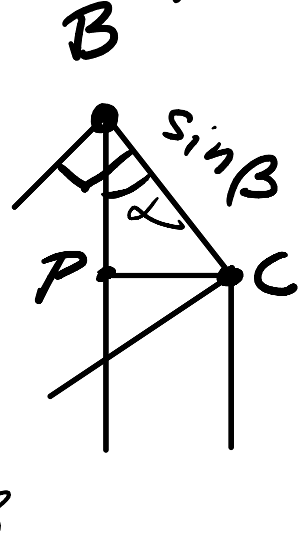
SHOW

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



I have made $\overline{AC}$ have length 1, not $\overline{AB}$. I have also made $\angle ABC$ a right angle, not $\angle ACB$. These changes took me a long time to stumble upon, even though they are right there in Figure E-1.5 on p. 19. $\overline{AD}$ is the length we are looking for, $\cos(\alpha-\beta)$. $\overline{AE}$ is $\cos\alpha\cos\beta$. Can we get the length of $\overline{ED}$?

Once again the top angle is α. So $\overline{PC}$ is $\sin\beta\sin\alpha$, and therefore so is $\overline{ED}$.

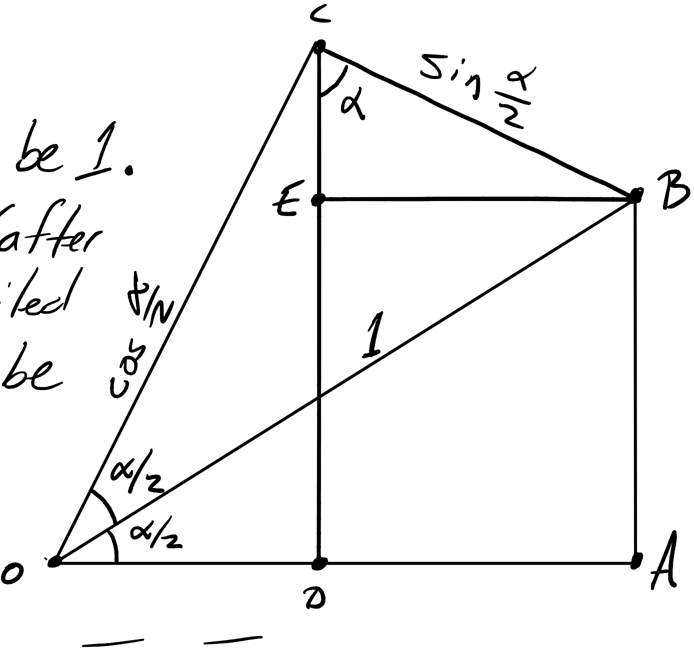


$$\overline{AD} = \overline{AE} + \overline{ED}$$

$$\Longrightarrow \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Problem 5

(a) Assign $\overline{OB}$ to be 1.
Our strategy (after many other failed attempts) will be to get two expressions for the length $\overline{CD}$.



$$\overline{CD} = \overline{CE} + \overline{ED} = \overline{CE} + \overline{AB}.$$

$\overline{OC}$ is $\cos\frac{\alpha}{2}$. So one expression for $\overline{CD}$ is $\cos\frac{\alpha}{2}\sin\alpha$.

$\overline{CB}$ is $\sin\frac{\alpha}{2}$.

$\angle ECB$ is α. So $\overline{CE}$ is $\sin\frac{\alpha}{2}\cos\alpha$.

$\overline{AB}$ is $\sin\frac{\alpha}{2}$. Put that into the equation:

$$\cos\frac{\alpha}{2}\sin\alpha = \sin\frac{\alpha}{2}\cos\alpha + \sin\frac{\alpha}{2}$$
$$= \sin\frac{\alpha}{2}(\cos\alpha + 1)$$

Now for the algebra. Square the equation.

$$\underbrace{\cos^2\frac{\alpha}{2}}_{{}^{*}\;1-\sin^2\frac{\alpha}{2}}\ \underbrace{\sin^2\alpha}_{1-\cos^2\alpha}{}^{**} = \sin^2\frac{\alpha}{2}(1+\cos\alpha)^2$$

$$1-\sin^2\frac{\alpha}{2} - \cos^2\alpha + \cancel{\sin^2\frac{\alpha}{2}\cos^2\alpha} = \sin^2\frac{\alpha}{2} + 2\cos\alpha\sin^2\frac{\alpha}{2} + \cancel{\cos^2\alpha\sin^2\frac{\alpha}{2}}$$

*, **: When proving trig identities, you should only use identities you have previously proved, or better yet none at all. So why have I let myself use these identities? Because the identity $\sin^2\theta + \cos^2\theta = 1$ comes straight from the Pythagorean Theorem, and that theorem is so fundamental, it is always fair game. →

However, since our job is to prove everything convincingly, maybe we should prove the Pythagorean Theorem too!? It is fabulous and not too hard.
SEE THE LAST PAGE

5(a) (CONT'D)

Get everything involving $\sin^2\frac{\alpha}{2}$ on the RHS and factor out the common factor of $2\sin^2\frac{\alpha}{2}$.

$$1-\cos^2\alpha = 2\sin^2\frac{\alpha}{2}(1+\cos\alpha)$$

Use $1-\cos^2\alpha = (1+\cos\alpha)(1-\cos\alpha)$. Cancel $1+\cos\alpha$ off of each side of the equation. (You can't cancel if $1+\cos\alpha = 0$, but that only happens if $\alpha = 180°$, and the formula we are trying to get works in that special case.)

$$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$$

Divide by 2 and take square root

$$\sqrt{\sin^2\frac{\alpha}{2}} = \sqrt{(1-\cos\alpha)/2}$$

If $\frac{\alpha}{2}$ is between 0° and 180°:***

$$\sqrt{\sin^2\frac{\alpha}{2}} = \sin\frac{\alpha}{2}$$

***: By the way, if $\alpha/2$ is between 180° and 360°, then what is $\sqrt{\sin^2\frac{\alpha}{2}}$?

So we have our formula

$$\sin\frac{\alpha}{2} = \sqrt{(1-\cos\alpha)/2}$$

and we know that it works for

$0 \le \alpha \le 180°$

Done with 5(a) ☺

Problem 5(b)

Prove the same thing starting with the cosine addition law:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Well, α and β can be anything, so it is fair to let $\beta=\alpha$. Then

$$\cos 2\alpha = \underbrace{\cos^2\alpha}_{1-\sin^2\alpha} - \sin^2\alpha = 1-2\sin^2\alpha$$

It is also fair to replace the variable α by $\alpha/2$ (everywhere!), leaving

$$\cos\alpha = 1-2\sin^2\frac{\alpha}{2}$$

$$2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$$

$$\sin^2\frac{\alpha}{2} = (1-\cos\alpha)/2$$

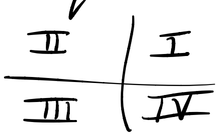
Take $\sqrt{\quad}$ of both sides

$$\sqrt{\sin^2\frac{\alpha}{2}} = \left|\sin\frac{\alpha}{2}\right|$$

$$\left|\sin\frac{\alpha}{2}\right| = \sqrt{(1-\cos\alpha)/2}$$

You can drop the absolute signs when $0 \le \frac{\alpha}{2} \le 180^\circ$ (e.g. when $\frac{\alpha}{2}$ is in quadrants I and II)

standard quadrant labeling →



Problem 6(a)

$$\sin 3\theta = \sin(2\theta+\theta)$$

$$\overset{*}{=} \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$$

$$= \underbrace{\sin(\theta+\theta)}_{**\ \sin\theta\cos\theta+\cos\theta\sin\theta}\cos\theta + \underbrace{\cos(\theta+\theta)}_{***\ \cos\theta\cos\theta-\sin\theta\sin\theta}\sin\theta$$

*, **, ***: I am not looking up any trig identities. I am just using the sine addition and cosine addition rules that we have already proved.

$$\sin 3\theta = \sin\theta\cos^2\theta + \cos^2\theta\sin\theta + \cos^2\theta\sin\theta - \sin^3\theta$$

$$= 3\underbrace{\cos^2\theta}_{1-\sin^2\theta}\sin\theta - \sin^3\theta$$

$$= 3\sin\theta - 4\sin^3\theta$$

Problem 6(b)

We start by putting $\theta = 1^\circ$ into the formula we just derived: $\sin 3^\circ = 3\sin 1^\circ - 4\sin^3 1^\circ$

The $\sin 3^\circ$ is a small number that we know.
Re-read p.11 if you don't remember how Ptolemy knew this number.

The number he wants and does not know is $\sin 1^\circ$.

Let's call $\sin 3^\circ$ ϵ.

Let's call $\sin 1^\circ$ x.

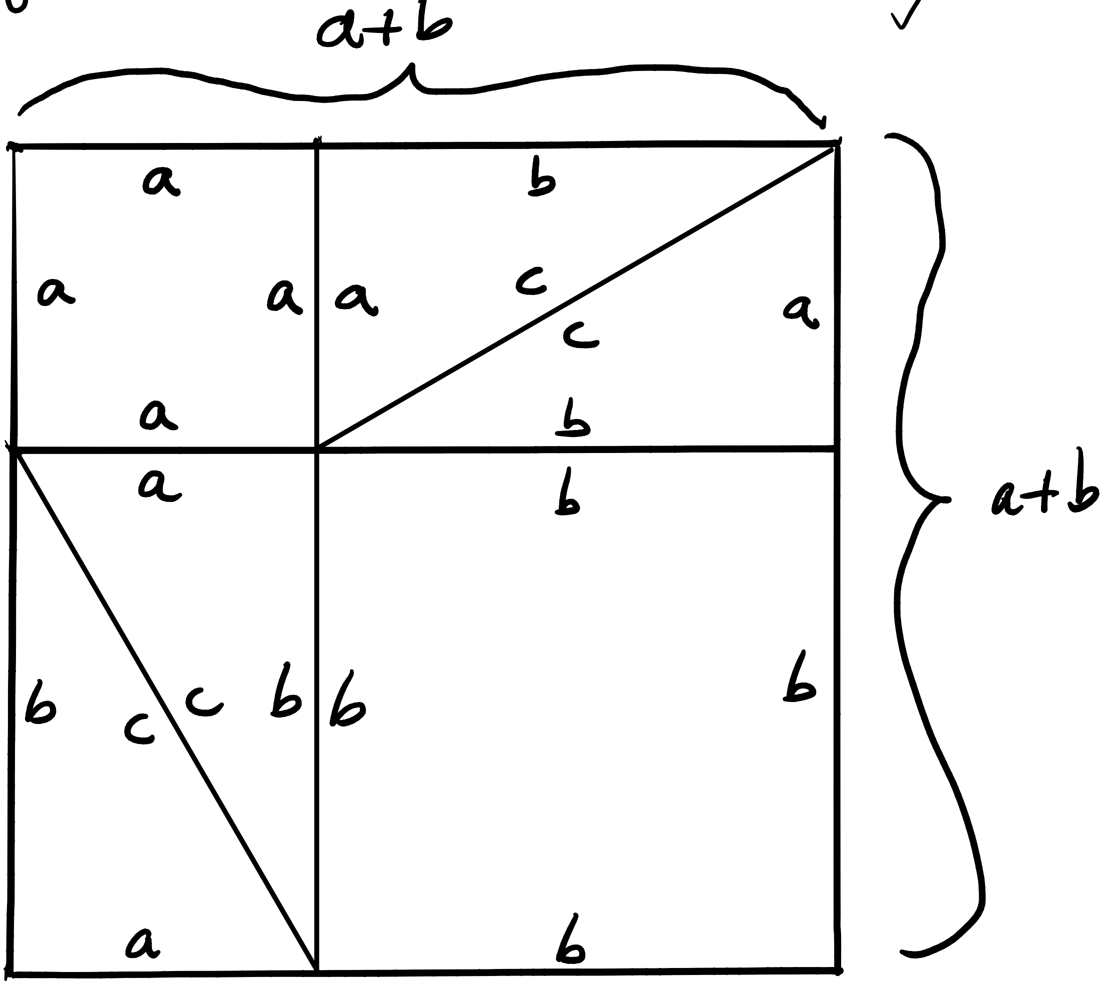
The formula is $\epsilon = 3x - 4x^3$

or $x = \dfrac{\epsilon + 4x^3}{3}$.

Van Brummelen suggests solving this equation to 100 decimal places. I will bust out Mathematica to do that. So the rest of 6(b) and 6(c) are in a separate PDF. Problem 10 we will do in-person.

Pythagorean Theorem

Neatly cut out the two squares and four triangles:



You agree that this whole thing has area $(a+b)^2$?

Using just the four triangles make something whose sides are also a+b. It will have a hollow space. What is the area of the hollow space? You have not used the two squares (whose area is a^2+b^2). What do you conclude?