Problem Set 4

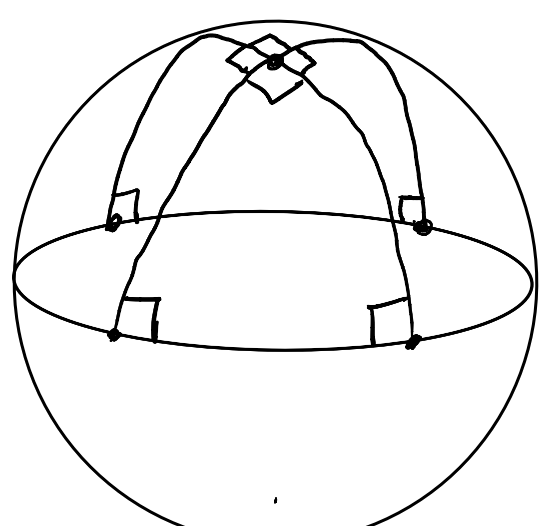
Since we did not get time to do it in class in addition to Problems 13 and 14 on p. 41, we first have Problem 10 on p. 40. Finally, this set of solutions closes with a proof of Menelaus's plane theorem on p. 45.

Problem 10 (The last problem of Problem Set 3)

Prove that if a spherical triangle has three right angles that it is its own polar triangle.

First, as a lemma, not as our entire proof, we observe that if you divide a hemisphere into four parts, each of which is therefore one-eighth of the entire sphere, then each of those four parts is a triangle with three right angles.

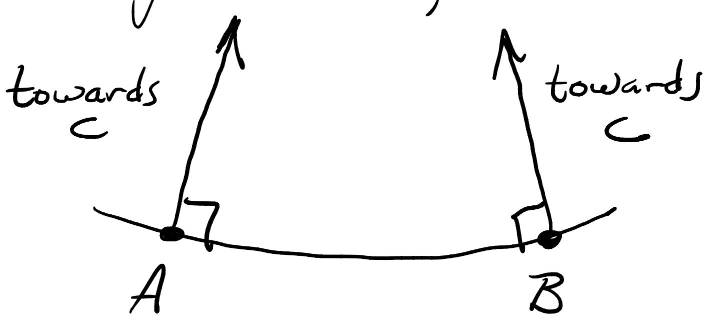
These are called octants. It is certainly the case that an octant is its own polar triangle.



Onward to our main proof...

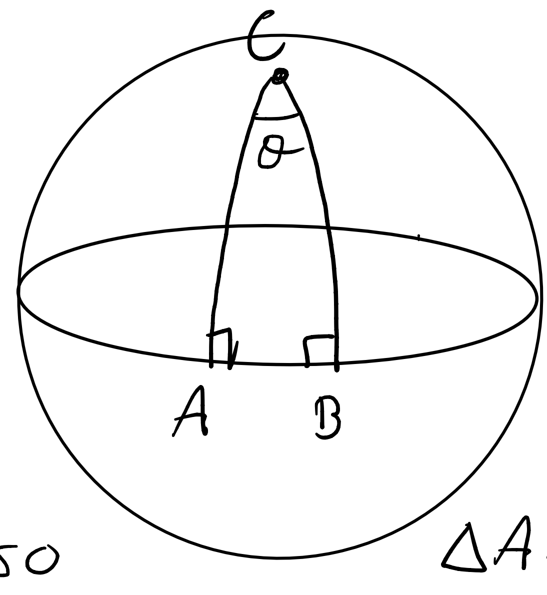
Problem 10 p.40 (CONT'D)

We begin by drawing one side of the triangle, AB, and noting that the sides leading toward C both form a 90° angle with AB (by assumption).

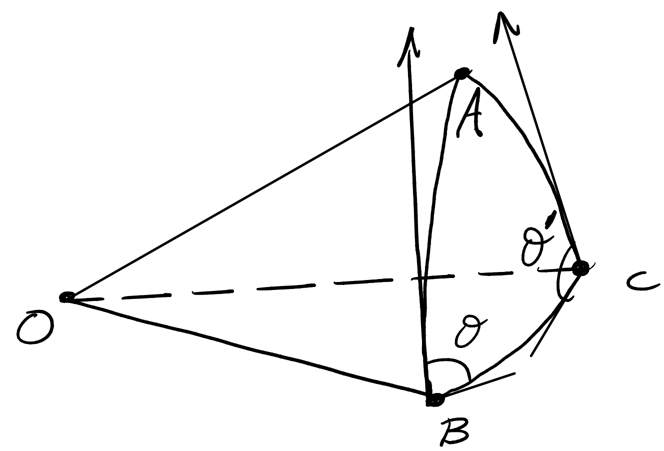


If we extend the side AB into a circle, we realize it can be thought of as an equator, and AC and BC are lines of longitude leading to a pole. In fact, the only place lines of longitude meet are at the poles, so C must be at a pole.

So so far we have established the picture at right. Of course, by assumption θ is also 90°, and that means ∠ABC fills 1/4 of a circle, so △ABC is indeed an octant. So we have shown that a triangle with three right angles must be an octant. And an octant is its own polar triangle.

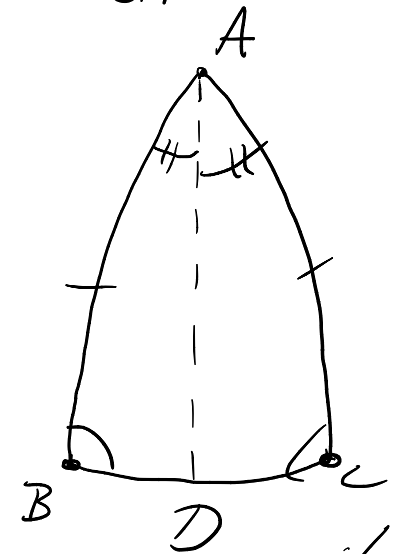


Problem 13 Show that a spherical triangle with two equal sides has two equal angles. Hint: Draw tangents to $\overset{\frown}{BA}$ at B and to $\widehat{CA}$ at A.



Frankly, I don't see what that buys us. We are trying to show that $\theta = \theta'$.

I would go a different direction. Look at ABC face on and bisect $\angle BAC$



If we mirror through the dashed line, because the bisected angles are the same and the lines $\overline{AB}$ and $\overline{AC}$ are the same, B must be at the mirror image position of C.
Hence $\triangle ABD$ is the mirror image of $\triangle ACD$.
Hence $\angle ABD = \angle ACD$

Problem 14

Let us call the three angles α, β, and γ. We are trying to show $\alpha - (\beta + \gamma) < 180°$.

Let us call the supplements of α, β, and γ δ, ϵ, and φ respectively. The Polar Duality Theorem tells us that $\delta, \epsilon, \varphi$ are the sides of the original triangle's polar triangle. The sides of any triangle must satisfy

$$\delta + \epsilon + \varphi < 360°$$

Now it is just algebra

$$\alpha - (\beta + \gamma) = 180° - \delta - (180° - \epsilon + 180° - \varphi)$$

$$= \epsilon + \varphi - \delta - 180°$$

$$= \delta + \epsilon + \varphi - 2\delta - 180°$$

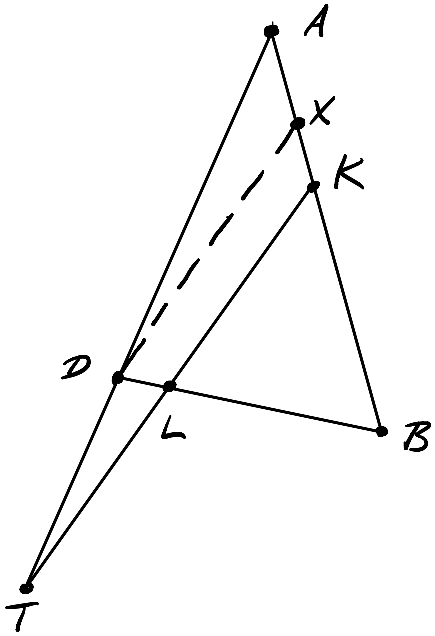
$$< 360° - 2\delta - 180° = 180° - 2\delta < 180°.$$

Quod Erat Demonstrandum

(Q.E.D.)

Re-Prove Menelaus's Plane Theorem (see p. 43)

First, we should re-draw the figure very asymmetrically so we have more sense of how the various sides scale.



Exactly as Van Brummelen has done, I have added DX parallel to TK.

Let's start with

$$\overline{AK} = \overline{AX} + \overline{XK} = \overline{AX} + \overline{BX} - \overline{BK} \quad (*)$$

why is that, perhaps, different and cleaver? Because the icchy unknowns $\overline{AX}$ and $\overline{BX}$ are whole sides (rather than parts of sides) of similar triangles.

Because $\triangle DAX$ is similar to $\triangle TAK$

$$\frac{\overline{AX}}{\overline{AK}} = \frac{\overline{AD}}{\overline{AT}} \quad \text{or} \quad \overline{AX} = \frac{\overline{AK}\cdot\overline{AD}}{\overline{AT}}$$

Because $\triangle DXB$ is similar to $\triangle LKB$

$$\frac{\overline{BX}}{\overline{BK}} = \frac{\overline{BD}}{\overline{LB}} \quad \text{or} \quad \overline{BX} = \frac{\overline{BK}\cdot\overline{BD}}{\overline{LB}}$$

On the next page, substitute for $\overline{AX}$ and $\overline{BX}$ in $*$.

Menelaus's Plane Theorem (CONT'D)

$$\overline{AK} = \overline{AX} + \overline{BX} - \overline{BK} = \frac{\overline{AK}\cdot\overline{AD}}{\overline{AT}} + \frac{\overline{BK}\cdot\overline{BD}}{\overline{LB}} - \overline{BK}$$

Now we look at where we are trying to get, and we see that Menelaus's Plane Theorem does not involve $\overline{AD}$ or $\overline{BD}$. That's fine because $\overline{AD} = \overline{AT} - \overline{TD}$ and $\overline{BD} = \overline{BL} + \overline{LD}$. Substitute,

$$\cancel{\overline{AK}} = \frac{\overline{AK}\cdot(\cancel{\overline{AT}} - \overline{TD})}{\overline{AT}} + \frac{\overline{BK}\cdot(\cancel{\overline{BL}} + \overline{LD})}{\overline{LB}} - \cancel{\overline{BK}}$$

Notice that some terms cancel, leaving,

$$0 = -\frac{\overline{AK}\cdot\overline{TD}}{\overline{AT}} + \frac{\overline{BK}\cdot\overline{LD}}{\overline{LB}}$$

Move the negative term to the LHS and divide through by $\overline{BK}$ $(=\overline{KB})$

$$\frac{\overline{AK}\cdot\overline{TD}}{\overline{KB}\cdot\overline{AT}} = \frac{\overline{LD}}{\overline{LB}}$$

Multiply through by $\frac{\overline{AT}}{\overline{TD}}$

$$\frac{\overline{AK}}{\overline{KB}} = \frac{\overline{AT}}{\overline{TD}}\cdot\frac{\overline{DL}}{\overline{LB}}$$

Quod Erat Demonstrandum