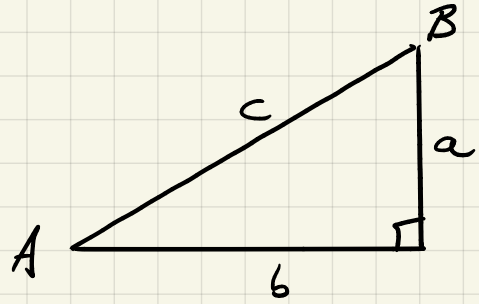
Problem Set 6 Solution

This problem set is a grand synthesis. Before grinding it out, let us contemplate what we are doing.

What is Trigonometry?



In its modern form, it is a bunch of statements about right triangles.

There are five quantities.

The statements are enumerable. The upshot is that given any two, you can usually get the other three.

Let us enumerate:

i.1	$b \tan A = a$	ii.1	$a = c \sin A$
i.2	$A + B = 90^\circ$	ii.2	$90^\circ - A = B$
i.3	$a \tan B = b$	ii.3	$90^\circ - B = A$
i.4	$c \cos A = b$	ii.4	$b = c \sin B$
i.5	$c \cos B = a$	ii.5	$c^2 = a^2 + b^2$

(ii.2 and ii.3: these are trivial rearrangements of i.2)

How do we use them? First getting sides:

To get	From	Use
a	b and c	ii.5
a	A and b	i.1
a	B and b	i.3
a	A and c	ii.1
a	B and c	i.5
a	A and B	not possible

To get	From	Use
b	a and c	ii.5
b	A and a	i.1
b	B and a	i.3
b	A and c	i.4
b	B and c	ii.4
b	A and B	not possible

To get	From	Use
c	a and b	ii.5
c	A and b	i.4
c	B and b	ii.4
c	A and a	ii.1
c	B and a	i.5
c	A and B	not possible

Note that if you only have two angles, you can never get a side. On the sphere we will be able to!

Now getting angles:

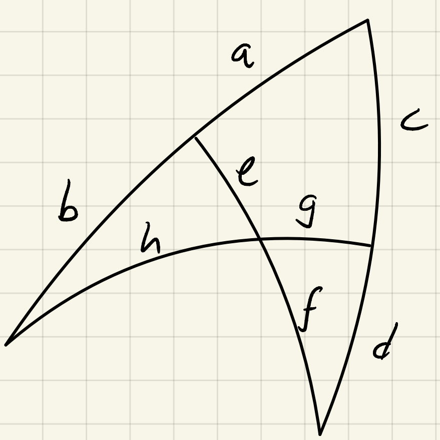
To get	From	Use
A	a and b	i.1
A	a and c	ii.1
A	b and c	i.4
A	a and B	ii.2 (or i.2 or ii.3)
A	b and B	ii.3 (or i.2 or ii.2)
A	c and B	i.2 (or ii.2 or ii.3)

To get	From	Use
B	a and b	i.3
B	a and c	i.5
B	b and c	ii.4
B	a and A	ii.2 (or i.2 or ii.3)
B	b and A	ii.3 (or i.2 or ii.2)
B	c and A	i.2 (or ii.2 or ii.3)

Note that if you have B you can get A or if you have A you can get B without needing a side.
On the sphere you will need a side!

This concludes "What is Trigonometry?" Onwards to the grand synthesis of Problem Set 6.

1(a) Use the Menelaus Theorems to get four Distinct Relationships among a, b, c, d, e, f, g in Figure 3.8 on p. 50.



From the bottom of p. 49, we have

$$D: \quad \frac{\sin a}{\sin b} = \frac{\sin(c+d)}{\sin d}\,\frac{\sin g}{\sin h}$$

$$C: \quad \frac{\sin(a+b)}{\sin a} = \frac{\sin(g+h)}{\sin g}\,\frac{\sin f}{\sin(e+f)}$$

$$MD: \quad \frac{\sin c}{\sin d} = \frac{\sin(a+b)}{\sin b}\,\frac{\sin e}{\sin f}$$

$$MC: \quad \frac{\sin(c+d)}{\sin c} = \frac{\sin(e+f)}{\sin e}\,\frac{\sin h}{\sin(g+h)}$$

Abbreviations
D = "Disjunction"
C = "Conjunction"
M = "Mirrored"

Hence
MD = "Mirrored Disjunction"
MC = "Mirrored Conjunction"

1(b) Simplify the four Expressions Assuming

$a+b=90^\circ \quad c+d=90^\circ \quad e+f=90^\circ \quad g+h=90^\circ$

In other words there are only four independent variables now, which we will take to be b, c, e, and h

can you already see why I have taken these four?

The first drastic simplification is that

$$\sin(a+b)=\sin 90^\circ=1 \qquad \sin(e+f)=\sin 90^\circ=1$$
$$\sin(c+d)=\sin 90^\circ=1 \qquad \sin(g+h)=\sin 90^\circ=1$$

Let's use those...

$$D: \quad \frac{\sin a}{\sin b}=\frac{1}{\sin d}\,\frac{\sin g}{\sin h}$$

$$C: \quad \frac{1}{\sin a}=\frac{1}{\sin g}\,\sin f$$

$$MD: \quad \frac{\sin c}{\sin d}=\frac{1}{\sin b}\,\frac{\sin e}{\sin f}$$

$$MC: \quad \frac{1}{\sin c}=\frac{1}{\sin e}\,\sin h$$

Now we use the assumptions again, but this time to get rid of a, d, f, and g...

$$a=90^\circ-b \quad \text{so} \quad \sin a=\sin(90^\circ-b)=\cos b$$
$$d=90^\circ-c \quad \text{so} \quad \sin c=\sin(90^\circ-d)=\cos c$$
$$f=90^\circ-e \quad \text{so} \quad \sin f=\sin(90^\circ-e)=\cos e$$
$$g=90^\circ-h \quad \text{so} \quad \sin g=\sin(90^\circ-h)=\cos h$$

Put all that in to D, C, MD, and DC...

D: $\dfrac{\cos b}{\sin b} = \dfrac{1}{\cos c}\,\dfrac{\cosh}{\sin h}$ ← Next use $\dfrac{\cos b}{\sin b} = \dfrac{1}{\tan b}$ and $\dfrac{\cos h}{\sin h} = \dfrac{1}{\tan h}$

C: $\dfrac{1}{\cos b} = \dfrac{1}{\cos h}\cos e$

MD: $\dfrac{\sin c}{\cos c} = \dfrac{1}{\sin b}\,\dfrac{\sin e}{\cos e}$ ↙ and use $\dfrac{\sin c}{\cos c} = \tan c$ and $\dfrac{\sin e}{\cos e} = \tan e$

MC: $\dfrac{1}{\sin c} = \dfrac{1}{\sin e}\sin h$

Note: I could have used $\dfrac{\cos b}{\sin b} = \cot b$ but I am not a fan of having both cot and tan kicking around.

So now we have

D: $\dfrac{1}{\tan b} = \dfrac{1}{\cos c}\,\dfrac{1}{\tan h}$

C: $\dfrac{1}{\cos b} = \dfrac{1}{\cos h}\cos e$

MD: $\tan c = \sin b \tan e$

MC: $\dfrac{1}{\sin c} = \dfrac{1}{\sin e}\sin h$

Get stuff out of denominators...

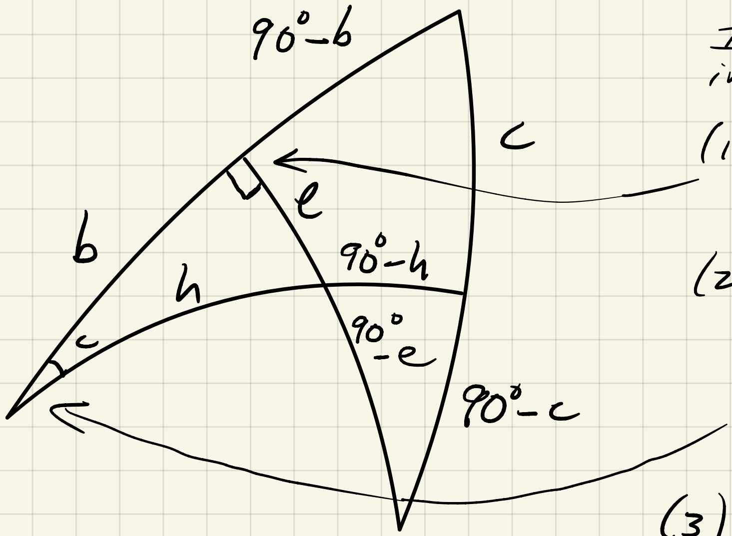
D: $\tan h \cos c = \tan b$

C: $\cos h = \cos b \cos e$

MD: $\tan c \sin b = \tan e$

MC: $\sin e = \sin c \sin h$

I should explain why I got rid of a, d, f, and g.
Redraw the Menelaus figure on the second page:



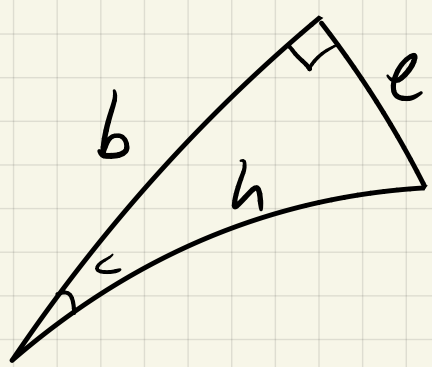
I have done three things in the re-drawing:

(1) I have noticed an important right angle

(2) I have noticed that the angle in the lower left is the same as the arc in the upper right

(3) I have gotten rid of a, d, f, and g.

Now we see why this is valuable!

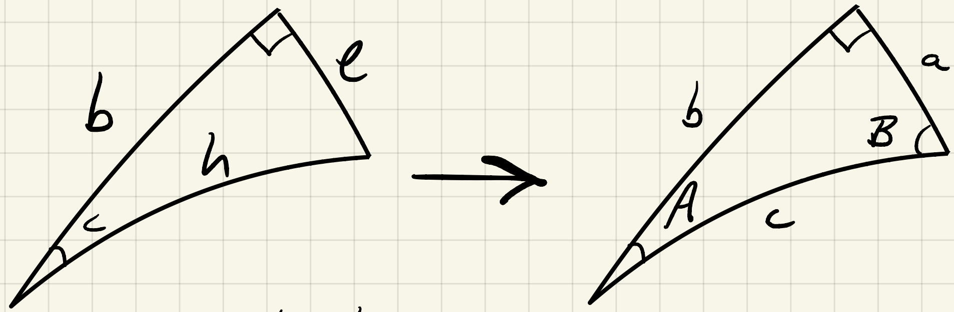


Although we started with a Menelaus diagram, all four of our identities are statements about just the right triangle in the lower left of the original, much more complex, figure.

Let us re-list them so that everything important about problem 1 is here on just the lower half of this page:

$$\tan h \cos c = \tan b$$
$$\cos h = \cos b \cos e$$
$$\tan c \sin b = \tan e$$
$$\sin e = \sin c \sin h$$

2. Rewrite what was learned in Problem 1 with modern right-triangle naming conventions.



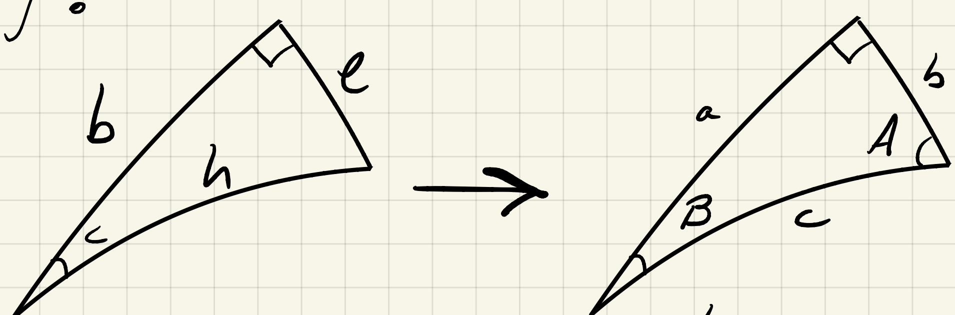
Our mapping is

$$\begin{aligned} b &\to b \\ e &\to a \\ h &\to c \\ c &\to A \end{aligned}$$

So

$$\begin{aligned}
\tan h \cos c &= \tan b \\
\cos h &= \cos b \cos e \\
\tan c \sin b &= \tan e \\
\sin e &= \sin c \sin h
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
\tan c \cos A &= \tan b \\
\cos c &= \cos b \cos a \\
\tan A \sin b &= \tan a \\
\sin a &= \sin A \sin c
\end{aligned}$$

Could we not have alternatively made this mapping?



In which case our mapping is

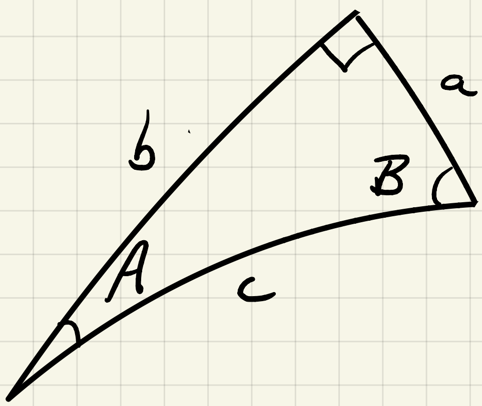
$$\begin{aligned} b &\to a \\ e &\to b \\ h &\to c \\ c &\to B \end{aligned}$$

So

$$\begin{aligned}
\tan h \cos c &= \tan b \\
\cos h &= \cos b \cos e \\
\tan c \sin b &= \tan e \\
\sin e &= \sin c \sin h
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
\tan c \cos B &= \tan a \\
\cos c &= \cos a \cos b \\
\tan B \sin a &= \tan b \\
\sin b &= \sin B \sin c
\end{aligned}$$

Wow, this equation is redundant (not new), but the other three seem new!

It is amazing (perhaps even disturbing) how much we have gotten by mirroring results. We have doubled and doubled and doubled the original Menelaus Theorem and our 1 result became 8 with only one being a duplicate. Let us re-list the 7 results so that everything important from Problem 2 is here on this page, and also number our results so that we can succintly refer to them...



$$\tan c \cos A = \tan b \quad (1)$$
$$\cos c = \cos b \cos a \quad (2)$$
$$\tan A \sin b = \tan a \quad (3)$$
$$\sin a = \sin A \sin c \quad (4)$$
$$\tan c \cos B = \tan a \quad (5)$$
$$\tan B \sin a = \tan b \quad (6)$$
$$\sin b = \sin B \sin c \quad (7)$$

3. Compare the 7 Identities Found in Problem 2 with the 10 Identities Van Brummelen Lists on p. 79

First I will list those. I don't like having both cotangents and tangents kicking around so I will take the recopying as an occasion to get rid of all the cotangents...

$\sin b \tan A = \tan a$ (I.1)	$\sin a = \sin c \sin A$ (II.1)
$\cos c \tan A \tan B = 1$ (I.2)	$\cos A = \cos a \sin B$ (II.2)
$\sin a \tan B = \tan b$ (I.3)	$\cos B = \cos b \sin A$ (II.3)
$\tan c \cos A = \tan b$ (I.4)	$\sin b = \sin c \sin B$ (II.4)
$\tan c \cos B = \tan a$ (I.5)	$\cos c = \cos a \cos b$ (II.5)

what equivalences are there?

(I.1) is (3)	(II.1) is (4)
(I.2) ?	(II.2) ?
(I.3) is (6)	(II.3) ?
(I.4) is (1)	(II.4) is (7)
(I.5) is (5)	(II.5 is (2)

But we are not done! We can use combinations of our 7 to get the remaining 3. You can get (I.2) from (2), (3), and (6). You can get (II.2) from (1), (2), and (7). And you can get (II.3) from (4), (5), and (2). This is just algebra, but for completeness I will do it on the next page.

Getting (I.2) from (2), (3), and (6).

$\cos c = \cos b \cos a \quad (2)$

$\tan A \sin b = \tan a \quad (3) \Rightarrow \tan A = \dfrac{\tan a}{\sin b}$

$\tan B \sin a = \tan b \quad (6) \Rightarrow \tan B = \dfrac{\tan b}{\sin a}$

Take the product $\cos c \tan A \tan B$ and substitute in what we have above...

$$\cos c \tan A \tan B = \cancel{\cos b}\, \cancel{\cos} a \frac{\cancel{\tan} a}{\cancel{\sin b}} \frac{\cancel{\tan b}}{\cancel{\sin} a} = 1$$

This is I.2

Getting (II.2) from (1), (2), and (7)

$\tan c \cos A = \tan b \quad (1) \Longrightarrow \cos A = \dfrac{\tan b}{\tan c}$

$\cos c = \cos b \cos a \quad (2)$

$\sin b = \sin B \sin c \quad (7) \Longrightarrow \sin B = \dfrac{\sin b}{\sin c}$

$$= \frac{\tan b}{\tan c} \frac{\cos b}{\cos c}$$

So,

$$\cos A = \sin B \frac{\cos c}{\cos b} = \cos a \sin B$$

This is II.2

Getting (II.3) from (4), (2), and (5)...

$\cos c = \cos b \cos a \quad (2)$

$\sin a = \sin A \sin c \quad (4)$

$\tan c \cos B = \tan a \quad (5) \Longrightarrow \cos B = \dfrac{\tan a}{\tan c}$

So,

$$\cos B = \frac{\tan a}{\tan c} = \frac{\sin a}{\sin c} \frac{\cos c}{\cos a} = \sin A \cos b$$

This is II.3

Epilog: What is Spherical Trigonometry?

In its modern form, it is a bunch of statements about right triangles on the sphere.

There are five quantities.

The statements about these quantities are enumerable.

The upshot is that given any two, you can <u>always</u> get the other three.

Let us compare the enumerable statements...

Planar Trigonometry

$b \tan A = a$	(i.1)	$a = c \sin A$	(ii.1)
$A + B = 90°$	(i.2)	$90° - A = B$	(ii.2)
$a \tan B = b$	(i.3)	$90° - B = A$	(ii.3)
$c \cos A = b$	(i.4)	$b = c \sin B$	(ii.4)
$c \cos B = a$	(i.5)	$c^2 = a^2 + b^2$	(ii.5)

(ii.2) and (ii.3): these are trivial rearrangements of (i.2)

Spherical Trigonometry

$\sin b \tan A = \tan a$	(I.1)	$\sin a = \sin c \sin A$	(II.1)
$\cos c \tan A \tan B = 1$	(I.2)	$\cos A = \cos a \sin B$	(II.2)
$\sin a \tan B = \tan b$	(I.3)	$\cos B = \cos b \sin A$	(II.3)
$\tan c \cos A = \tan b$	(I.4)	$\sin b = \sin c \sin B$	(II.4)
$\tan c \cos B = \tan a$	(I.5)	$\cos c = \cos a \cos b$	(II.5)

The planar and spherical equations are deeply related. The relations are most easily seen when the ordering of the equations are as shown. Nonetheless, this is a whole 'nother discussion. Perhaps we will have time for it in class. Van Brummelen gets into it on p.79 in a section titled, "Applying the Locality Principle."

One more good exercise, left to you, is to make tables of how we use the spherical trigonometry equations to get any quantity from any two of the others.

As an example:

To get	From	Use
a	b and c	II.5

29 table rows to go!