Problem Set 8 — Solution

Due Thursday, April 7

Problem 1 — Summer Solstice

The summer solstice is the longest day of the year.

(a) How much daylight is there at Deep Springs on the summer solstice?

The summer solstice occurs when the ecliptic longitude of Sun $\lambda = 90°$. On that day, $\delta = \epsilon = 23.44°$ and $\alpha = 90°$.

The usual formula for η, the ortive amplitude is $\sin\eta = \sin\delta / \cos\phi$.

```
In[∘]:= η = ArcSin[Sin[23.44 Degree] / Cos[37.4 Degree]] / Degree
Out[∘]= 30.0484
```

We can then get the ascensional difference (or "equation of daylight") from $\cos n \cos\delta = \cos\eta$.

```
In[∘]:= n = ArcCos[Cos[η Degree] / Cos[23.44 Degree]] / Degree
Out[∘]= 19.3591
```

We convert this to hours, multiply by 2 and add it to 12, and we have the amount of daylight:

```
In[∘]:= 12 + 2 n / 15
Out[∘]= 14.5812
```

This is 14 h 35 m. If you go to www.timeanddate.com, they say sunrise is 5:33 am and sunset is 8:17pm. So they get 14 hours and 44 minutes. The extra 9 minutes is caused by factors we discussed in class.

(b) Where along the horizon will the sun rise and set at Deep Springs on the summer solstice?

That is what η tells us. The answer is 30.0 degrees north of east for sunrise, and 30.0 degrees north of west for sunset.

(c) Repeat for Seattle, latitude 47.6°.

```
In[∘]:= ηSeattle = ArcSin[Sin[23.44 Degree] / Cos[47.6 Degree]] / Degree
Out[∘]= 36.1518
```

```
In[∘]:= nSeattle = ArcCos[Cos[ηSeattle Degree] / Cos[23.44 Degree]] / Degree
Out[∘]= 28.3475
```

In[•]:=
```
12 + 2 nSeattle / 15
```
Out[•]=
```
15.7797
```

So Seattle gets 15h 47m of daylight, and the Sun rises 36.1° degrees north of east, and sets 36.1° north of west.

Problem 2 — Winter Solstice

Repeat all of Problem 1 for the Winter Solstice

First Deep Springs:

(a) The longitude of Sun is $\lambda = 270°$. On that day, $\delta = -\epsilon = -23.44°$ and $\alpha = 270°$.

In[•]:=
```
12 - 2 n / 15
```
Out[•]=
```
9.41879
```

This is 9 h 25m.

(b) We still just need η. The answer is 30.0 degrees south of east for sunrise, and 30.0 degrees south of west for sunset.

(c) Repeat for Seattle, latitude 47.6°.

In[•]:=
```
12 - 2 nSeattle / 15
```
Out[•]=
```
8.22034
```

So Seattle gets 8h 13m of daylight, and the Sun rises 36.1° degrees south of east, and sets 36.1° south of west.

Problem 3 — Arcturus

The star Arcturus, part of the constellation Boötes, is the brightest start in the northern hemisphere of the celestial sphere.

(a) How long is Arcturus visible in the sky each day? Does it change?

To answer this, we need the right ascension and declination of Arcturus, which is easy to look up. Actually, we only need the declination for part (a). Anyway they are right ascension 14h 15m 40s, declination +19° 10’ 56”.

In[•]:=
```
αArcturus = N[(14 + 15 / 60 + 40 / 3600) * 15]
δArcturus = N[19 + 10 / 60 + 56 / 3600]
```
Out[•]= 213.917

Out[•]= 19.1822

In[•]:=
```
ηArcturus = ArcSin[Sin[δArcturus Degree] / Cos[37.4 Degree]] / Degree
```
Out[•]= 24.4315

In[•]:=
```
nArcturus = ArcCos[Cos[ηArcturus Degree] / Cos[δArcturus Degree]] / Degree
```
Out[•]= 15.4253

In[•]:=
```
12 + 2 nArcturus / 15
```
Out[•]= 14.0567

This is 14h 3m. It does not change because it only depends on the star's right ascension and declination, and these do not change.

(b) When will Arcturus appear on the horizon relative to Sunset today.

Let's take "today" to mean the date the problems were distributed, March 31, 2022.

According to Ptolemy's tables $\lambda = 10.6^o$ on March 31st. Therefore,

In[•]:=
```
λ = 10.6;
ϵ = 23.44;
δSun = ArcSin[Sin[λ Degree] Sin[ϵ Degree]] / Degree
αSun = ArcCos[Cos[λ Degree] / Cos[δSun Degree]] / Degree
```
Out[•]= 4.1963

Out[•]= 9.74275

So Sun's declination is 4.20^o and its right ascension is 0h 39m.

Now to bring this mess home. If both Sun and Arcturus had zero declination, then the amount that Arcturus trails Sun is:

In[•]:=
```
αArcturus - αSun
```
Out[•]= 204.174

However, due to Sun's declination, it sets later by nSun where

```
In[•]:= ηSun = ArcSin[Sin[δSun Degree] / Cos[37.4 Degree]] / Degree;
        nSun = ArcCos[Cos[ηSun Degree] / Cos[δSun Degree]] / Degree;
        nSun
Out[•]= 3.21575
```

Meanwhile, due to Arcturus' declination, it rises earlier by nArcturus where

```
In[•]:= ηArcturus = ArcSin[Sin[δArcturus Degree] / Cos[37.4 Degree]] / Degree;
        nArcturus = ArcCos[Cos[ηArcturus Degree] / Cos[δArcturus Degree]] / Degree;
        nArcturus
Out[•]= 15.4253
```

Both of these conspire to reduce the amount that the rise of Arcturus trails the setting of the Sun. There is one more effect. We are comparing a rise with a set. This is an additional 12 hours of reduction.

```
In[•]:= (αArcturus - αSun - nSun - nArcturus) / 15 - 12
Out[•]= 0.36886
```

Our answer is 22m. SkySafari says sunset is at 7:14 and Arcturus rises at 7:26, which is only 12 minutes apart. I think we can attribute this to the usual two effects: (i) The Sun's disk is substantial, delaying sunset a bit, and there is atmospheric refraction, delaying sunset, and making the rise of Arcturus sooner.

Problem 4 — Mizar

The star Mizar is in the Ursa Major constellation (the Big Bear of which the Big Dipper is the most prominent part). What happens if you try to calculate the ortive amplitude or the equation of daylight of Mizar using the spherical trig identities? Why?

```
In[•]:= αMizar = N[(13 + 23 / 60 + 55 / 3600) * 15]
        δMizar = N[54 + 55 / 60 + 38 / 3600]
Out[•]= 200.979
Out[•]= 54.9272

In[•]:= ηMizar = ArcSin[Sin[αMizar Degree] / Cos[37.4 Degree]] / Degree
Out[•]= -26.7875

In[•]:= nMizar = ArcCos[Cos[ηMizar Degree] / Cos[δMizar Degree]] / Degree
Out[•]= 0. + 57.8023 i
```

Mathematica gives an imaginary number for the equation of daylight. This is because Mizar is well within 37.4° of the celestial north pole. Therefore it never rises and sets. Most calculators will display NaN (for not a number) or Error.

Problem 5 — Home Town

Which direction is your home town from Deep Springs?

This is a Law of Cosines problem now that we have that tool.

In[◦]:=
```
ϕDeepSprings = 37.3717;
λDeepSprings = -117.9842;
ϕNanaimo = 49.1659;
λNanaimo = -123.9401;

dλ = (λDeepSprings - λNanaimo)
```

Out[◦]= 5.9559

The triangle we form has 90° - ϕNanaimo as one, side 90° - λNanaimo as another side, and dλ as the angle between those two sides. From these we get the distance from Deep Springs to Nanaimo.

In[◦]:=
```
d = ArcCos[Cos[90 Degree - ϕDeepSprings Degree] Cos[90 Degree - ϕNanaimo Degree] +
     Sin[90 Degree - ϕDeepSprings Degree]
      Sin[90 Degree - ϕNanaimo Degree] Cos[dλ Degree]] / Degree
```

Out[◦]= 12.5562

Then we use the Law of Sines to get the direction.

In[◦]:=
```
a = ArcSin[Sin[90 Degree - ϕNanaimo Degree] Sin[dλ Degree] / Sin[d Degree]] / Degree
```

Out[◦]= 18.1852

That is how much west of north I need to go. The distance d is 753 nautical miles or 867 statute miles.

Problems 6 (Optional) — Mecca from Deep Springs

What direction is Mecca from Deep Springs?

In[◦]:=
```
ϕMecca = 21.3891;
λMecca = 39.8579;

dλMecca = (λMecca - λDeepSprings)
```

Out[◦]= 157.842

In[◦]:=
```
dMecca = ArcCos[Cos[90 Degree - ϕDeepSprings Degree] Cos[90 Degree - ϕMecca Degree] +
     Sin[90 Degree - ϕDeepSprings Degree]
      Sin[90 Degree - ϕMecca Degree] Cos[dλMecca Degree]] / Degree
```

Out[◦]= 117.643

```
In[•]:= aMecca = ArcSin[
           Sin[90 Degree - ϕMecca Degree] Sin[dλMecca Degree] / Sin[dMecca Degree]] / Degree
Out[•]= 23.3555
```

Problems 7 (Optional) — LAS to ORD

This was to make up your own problem. I am interested in why flying from LAS to ORD (which got made into a Term 5 exam problem) has no solution if you accidentally put in the wrong value for the initial direction of the flight. I am interested because the wrong value almost got put into the exam. Acckk! Here is the problem:

Las Vegas (LAS) has latitude 36°. Chicago O'Hare (ORD) has latitude 42°. When a plane takes off on a great circle route from Las Vegas headed to O'Hare, its initial compass heading is 72° (THIS IS THE WRONG VALUE THAT ALMOST GOT ONTO THE EXAM). What is the plane's compass heading upon arrival into O'Hare?

The solution is a straightforward application of the Law of Sines.

$\sin(?) / \sin(90^\circ - 36^\circ) = \sin(72^\circ) / \sin(90^\circ - 42^\circ)$

```
In[•]:= N[ArcSin[Sin[72 Degree] Sin[54 Degree] / Sin[48 Degree]] / Degree]
Out[•]= 90. - 15.1918 ⅈ
```

As you can see, Mathematica has given us an imaginary number, which as in Problem 4 is its way of telling us there is no triangle that can work.

The right value, that works just fine, is 66°.

```
In[•]:= N[ArcSin[Sin[66 Degree] Sin[54 Degree] / Sin[48 Degree]] / Degree]
Out[•]= 84.
```

(This is the direction from which the plane arrives at ORD. You need to take 180° - 84° = 96° to get the compass heading upon arrival.)

How can the small difference between 66° and 72° cause there to be no solution. Drawing the whole thing out on a Lenart sphere is the only way to clearly see it, and sure enough, the drawing on the sphere shows that the 72° takeoff line never crosses the ORD line of latitude.

Problem 8

Van Brummelen p. 92 #6.

(a) Is there a right spherical triangle with B = 100° (big) but b = 30° (small?).

Well maybe, and if so we can compute its other sides. By 1.3, II.3, and II.4 and choosing different values of the arcsin when necessary so that we don't get negative numbers

```
In[∘]:= B = 100;
        b = 30;
        a = 180 - N[ArcSin[Tan[b Degree] / Tan[B Degree]] / Degree]
Out[∘]= 185.843
```

```
In[∘]:= A = 180 - N[ArcSin[Cos[B Degree] / Cos[b Degree]] / Degree]
Out[∘]= 191.567
```

```
In[∘]:= c = N[ArcSin[Sin[b] / Sin[B]]]
Out[∘]= 1.5708 - 1.28833 i
```

Now we have foundered. The equation for c tells us there is no solution. Sigh.

(b) Show that no isosceles right spherical triangle can have its hypotenuse greater than 90° nor its acute angle less than 45°.

Well, by isosceles, we mean that a = b and A = B.

By II.5, cos c = cos a cos b, but since $a = b$, $\cos c = \cos^2 a$. The right hand side is always non-negative. But if c > 90° then the left hand side is negative.

The other thing we are supposed to show is that its acute angle must not be less than 45°. By I.2,

cos c = cot A cot B, but since A = B, cos c = $\cot^2$A. But if A > 45°, cot A > 1, and cos c cannot be greater than 1.

Problem 9

Van Brummelen p. 92 #8.

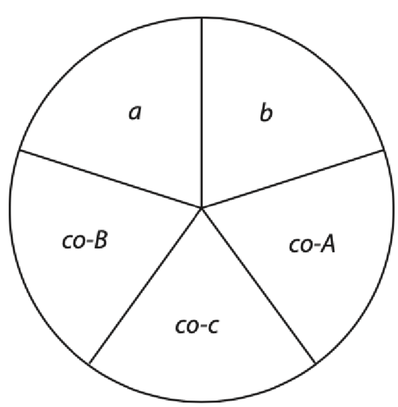
A quadrantal triangle has one of its sides (not one of its angles) equal to 90°.

(a) In general, how might the identities of Napier's rules be used to solve quadrantal triangles?

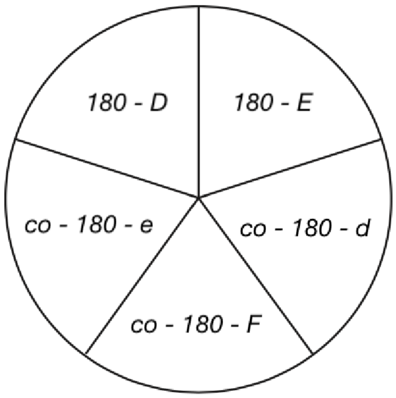
We can call the sides d, e, and f, with f = 90°, and the angles D, E, and F. The polar triangle to DEF has sides 180-D, 180-E, and 180-F, and angles 180-d, 180-e, and 180-f = 180-90 = 90. So identifying:

a = 180-D
b = 180-E
c = 180-F
A = 180-d
B = 180-e

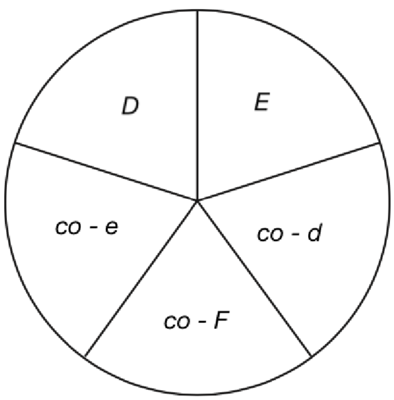
we can turn Napier's Rules into statements about D, E, F, d, and e. First we need to translate the diagram:



It becomes:



Then we have to do something about all the 180’s, but that is easy because sin(180-θ) is just sin θ whereas cosine, tangent, and cotangent all satisfy fun(180-θ) = -fun θ.



Napier’s Rule I for Quadrantal Triangles: The sine of any circular part is equal to the product of the tangents of the two parts adjacent to it. If there are zero or two sines in the resulting equation, the equation gets a minus sign.

Napier’s Rule for II Quadrantal Triangles: The sine of any circular part is equal to the product of the cosines of the two parts opposite to it. If there are zero or two sines in the resulting equation, the equation gets a minus sign.

(b) Solve the triangle D = 69°, F = 78°.

sin D = sin F sin d
sin e = tan D cot F
cos F = - cos D cos E

```
In[•]:= angleD = 69;
        angleF = 78;
        d = N[ ArcSin[Sin[angleD Degree] / Sin [angleF Degree]] / Degree]
Out[•]= 72.6378

In[•]:= e = N[ ArcSin[Tan[angleD Degree] / Tan [angleF Degree]] / Degree]
Out[•]= 33.6232

In[•]:= angleE = N[ ArcCos[-Cos[angleF Degree] / Cos [angleD Degree]] / Degree]
Out[•]= 125.462
```

(c) Explain why a spherical triangle with three right angles must have all three sides equal to 90° as well.

By Rule II for Quadrantal Triangles, sin E = sin e sin F. Since sin E and sin F are 1, sin e must be 1. In other words, e must be 90°. This of course applies to all three sides.