Manhattan Project — Assignment 3 — Half Lives and Decays

Let’s recap the equations on p. 26 before doing any problems.

Decay Rate Derivation

I derived the equation for decay rate without using any calculus. I had to use some properties of the exponential though. One of the properties was this one that you may or may not be familiar with:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

If you are curious and know factorials, the denominators are 0!=1, 1!=1, 2!=2, 3!=6, etc. I will use the above formula to make an approximation below. Another property of the exponential that I used is:

$$e^{x+y} = e^x\, e^y$$

That’s actually true whatever base is being exponentiated. For example:

$$10^{x+y} = 10^x \cdot 10^y$$

Finally, I used $e^0 = 1$. At least I'm not using any calculus!

So we start with the claim that

$$N(t) = N_0\, e^{-\lambda t}$$

describes radioactive decay. If you have N_0 atoms at time $t = 0$, this formula is the one that tells you how much you have at any later time.So it certainly tells you how much you have at both time t and time $t + \Delta t$ where Δt is a small amount of time. We have:

$$N(t + \Delta t) = N_0\, e^{-\lambda(t+\Delta t)} = N_0\, e^{-\lambda t}\, e^{-\lambda \Delta t} = N(t)\, e^{-\lambda \Delta t} = N(t)\left[1 - \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2} - \frac{(\lambda \Delta t)^3}{6} + \ldots\right]$$

Here comes the tricky part! If $\lambda \Delta t$ is small (think of something like 0.01), then every term in the infinite series is 0.01 times as small as the previous one. Let's neglect all but the first two!

$$N(t + \Delta t) = N(t)[1 - \lambda \Delta t] = N(t) - \lambda \Delta t N(t)$$

Rearrange:

$$N(t + \Delta t) - N(t) = -\lambda \Delta t N(t)$$

Rearrange more:

$$\frac{N(t+\Delta t)-N(t)}{\Delta t} = -\lambda N(t)$$

What we have on the left side is what Reed calls $R(t)$ in equation 2.3. It is the rate that the number of particles is changing. The right-hand-side of the equation has a minus sign because the number of particles is decreasing. Let's summarize before we move on:

$$R(t) = -\lambda N(t)$$

Relationship Between λ and $t_{1/2}$

There is another thing we derived in class that I want to re-derive here: the relationship between λ and $t_{1/2}$.

$t_{1/2}$ is the time at which you have half as many particles. So on the left-hand side of,

$$N(t) = N_0\, e^{-\lambda t}$$

we put $N_0/2$ and for t on the right-hand side, we put $t_{1/2}$,

$$N_0/2 = N_0\, e^{-\lambda t_{1/2}}$$

The N_0 on each side cancels, leaving:

$$1/2 = e^{-\lambda t_{1/2}}$$

Now take the reciprocal of each side of the equation:

$$2 = e^{\lambda t_{1/2}}$$

Finally take the natural log of each side. The natural log is by definition the function that undoes the exponential:

$$\ln 2 = \lambda t_{1/2}$$

We have Reed’s equation 2.2:

$$\lambda = \frac{\ln 2}{t_{1/2}}$$

1. Using $R(t) = -\lambda N(t)$ and $\lambda = \frac{\ln 2}{t_{1/2}}$

(a) Convert 138 days to seconds.

(b) If you start off with an Avogadro's number of Polonium-210 atoms $\left(N_A \approx 6.02 \cdot 10^{23}\right)$ and the half life of Polonium-210 is $t_{1/2} = 138$ days, what number of atoms will be decaying per second.

(c) A Curie (abbreviated Ci) is $3.7 \cdot 10^{10}$ decays / second. Convert your answer in (b) to Ci.

2. Alpha Decay

Polonium-210 alpha decays. The reaction is:

$${}^{210}_{84}\text{Po} \to {}^{4}_{2}\text{He} + {}^{A}_{Z}\text{X}$$

(a) What must A and Z be?

(b) Use the Table of Isotopic Masses and Natural Abundances you have. What element has the Z you found in (a)?

3. β^- and β^+ Decay

(a) Suppose Polonium-210 did a β^- decay. Consult Fig. 2.12 to find out what the N and Z value of the resulting nucleus would be. (N is the number of neutrons and $N = A - Z$.)

$${}^{210}_{84}\text{Po} \to e^- + {}^{A}_{Z}\text{X}$$

In addition to reporting N and Z of the new nucleus, what is the A value of the new nucleus?

(b) Suppose Polonium-210 did a β^+ decay. Consult Fig. 2.12 to find out what the N and Z value of the resulting nucleus would be.

$${}^{210}_{84}\text{Po} \to e^+ + {}^{A}_{Z}\text{X}$$

In addition to reporting N and Z of the new nucleus, what is the A value of the new nucleus?

4. Energy Released in Fission

Returning to the actual Polonium-210 alpha decay that you found in Problem 2:

$^{210}_{84}\text{Po} \rightarrow \ ^{4}_{2}\text{He} + ^{A}_{Z}\text{X}$

Look up the mass of each atom involved in your the Table of Isotopic Masses and Natural Abundances. Actually, Polonium-210 isn't stable enough to be in our table, so I'll just tell you that it has mass 209.982874u.

(a) What is the total mass on the left-hand side. This is super-easy! There is only one reactant on the left-hand side.

(b) What is the total mass on the right-hand side. Keep all six decimal places.

(c) What is the difference?

DISCUSSION: Notice that when you compute the difference, you are down to four significant figures even though started with seven significant figures for Helium-4 and nine significant figures for the other nuclei.

I give a bunch of exact values below. If you want to round the result of any step to four significant figures, you can.

(d) Using 1amu (or 1u) = 1.66054×10^{-27}kg and multiply by c^2 to convert what you got in (a) to Joules. For some additional accuracy, let's use $c = 2.99792458 \cdot 10^8$ m/s instead of $c = 3 \cdot 10^8$ m/s.

(e) Using 1eV = $1.602176634 \cdot 10^{-19}$J, convert what you got in (d) to eV.

DISCUSSION: Almost comically, since 1993 that value of $c = 2.99792458 \cdot 10^8$ m/s is exact, just like since 2019 the value of $N_A = 6.02214076 \cdot 10^{23}$ and the value of 1eV = $1.602176634 \cdot 10^{-19}$ J are both exact. It would be a fun detour to discuss why all these values are now exact values.

(f) Using 1MeV is 10^6eV, convert what you got in (e) to MeV.

(g) Steps (d), (e), and (f), are just conversions that always involve the same steps, and it gets tiring doing them over and over again. At the middle of p. 34, Reed quotes the conversion factor for atomic mass units to MeV. Use that conversion factor to go straight from (c) to (f) in a single step. You should get something very close.