Manhattan Project - Assignment 5 - Solution

1. The Radius of the Uranium Nucleus

The empirical formula on p. 50 is

$$\text{radius} \sim a_0 A^{1/3} \quad \text{where} \quad a_0 \sim 1.2 \times 10^{-15}\,m$$

For U-238, A = 238. Putting that into a calculator,

$$\text{radius} \sim 1.2\,fm \times 238^{1/3} = 7.4\,fm$$

I meant to also ask, what cross-section does that give?

cross-section (naively) = area of disk = πr^2

$$= \pi \cdot (7.4\,fm)^2 = 173.7\,fm^2 = 1.7\,\text{barns}$$

2. The Surface Area of the Uranium Nucleus

(a) $A = 4\pi R^2 = 4\pi(7.4\,fm)^2 = 688\,fm^2$

(b) $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi (7.4\,fm)^3 = 1697\,fm^3$

3. Reed Problem 3.3 p.120

$$U_{self} = \underbrace{\frac{3e^2}{20\pi\epsilon_0 a_0}}_{\text{This mess is } \frac{3}{5} \text{ of } \frac{e^2}{4\pi\epsilon_0 a_0}} \frac{Z^2}{A^{1/3}}$$

and that mess was given as 1.2 MeV back on p. 50

$$\frac{3}{5}\,1.2\,MeV = \frac{3.6}{5}\,MeV = 0.72\,MeV$$

3 again

If you don't like that I used Eq. 2.28 on p. 50, then you can do it all from scratch:

$$e = 1.6\times10^{-19}\,C$$
$$a_0 = 1.2\times10^{-15}\,m$$
$$\epsilon_0 = 8.85\times10^{-12}\,C^2/Nm^2$$

And the conversion factor from Joules to MeV is

$$1\,MeV = 1.6\times10^{-13}\,J$$

So

$$\frac{e^2}{4\pi\epsilon_0 a_0} = \frac{(1.6\times10^{-19}\,\cancel{C})^2}{4\cdot\pi\cdot 8.85\times\frac{10^{-12}\,\cancel{C^2}}{N\cancel{m^2}}\cdot 1.2\times10^{-15}\,\cancel{m}}$$

$$= \underbrace{10^{-38+12+15}}_{10^{-11}} \cdot \underbrace{\frac{1.6}{4\cdot\pi\cdot 8.85\cdot 1.2}}_{0.019}\ \underbrace{Nm}_{J}$$

Ok, but this is usually quoted in MeV, not Joules, so

$$\frac{e^2}{4\pi\epsilon_0 a_0} = 0.019\times10^{-11}\,J \times \frac{1\,MeV}{1.6\times10^{-13}\,J}$$

$$= 0.012\times10^2\,MeV = 1.2\,MeV$$

Then

$$U_{self} = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 a_0}\frac{Z^2}{A^{1/3}}$$

$$= \frac{3}{5}\times 1.2\,MeV\,\frac{Z^2}{A^{1/3}}$$

$$= 0.72\,MeV\,\frac{Z^2}{A^{1/3}}$$

4. Reed Problem 3.4 p.120

Before there is just one sphere with Z protons and A nucleons

Its self-energy is

$$U_{self,\,before} = \frac{3e^2}{20\pi\epsilon_0 a_0}\,\frac{Z^2}{A^{1/3}}$$

After there are two spheres with $Z/2$ protons and $A/2$ nucleons

so

$$U_{self,\,after} = 2\cdot\frac{3e^2}{20\pi\epsilon_0 a_0}\,\frac{(Z/2)^2}{(A/2)^{1/3}}$$

Also after, we have the Coulomb barrier of the two touching spheres.

$$U_{Coulomb} = \frac{e^2}{4\pi\epsilon_0 a_0}\,\frac{(Z/2)^2}{(A/2)^{1/3}+(A/2)^{1/3}}$$

The whole point is that the

$$U_{self,\,before} > U_{self,\,after} + U_{Coulomb}$$

and this energy is available to become (violent) kinetic energy. Let's calculate the difference.

kinetic energy = KE

4. (CONT'D)

$$KE = U_{self,\,before} - U_{self,\,after} - U_{coulomb}$$

$$= \frac{e^2}{4\pi\epsilon_0 a_0}\left(\frac{3}{5} - 2\cdot\frac{3}{5}\,\frac{(1/2)^2}{(1/2)^{1/3}} - \frac{(1/2)^2}{2\,(1/2)^{1/3}}\right)\frac{Z^2}{A^{1/3}}$$

It seems that Reed wants us to put the $\frac{3}{5}$ in with the $\frac{e^2}{4\pi\epsilon_0 a_0}$ and call that $a_c = \frac{3e^2}{20\pi\epsilon_0 a_0}$.

and in the previous problem we computed $a_c = 0.72\,\mathrm{MeV}$.

$$KE = a_c\left(1 - 2\,(1/2)^{5/3} - \frac{5}{3}\,\frac{1}{2}\,(1/2)^{5/3}\right)\frac{Z^2}{A^{1/3}}$$

$$= a_c\left(1 - \frac{17}{6}\,(1/2)^{5/3}\right)\frac{Z^2}{A^{1/3}}$$

$$= a_c\left(1 - \frac{17}{12}\,(1/2)^{2/3}\right)\frac{Z^2}{A^{1/3}}$$

$2+\frac{5}{6} = \frac{17}{6}$

Put in $Z=92$, $A=235$, $a_c=0.72\,\mathrm{MeV}$, and get $KE = 106\,\mathrm{MeV}$. In 3.2 Reed says that the actually measured value is $185.7\,\mathrm{MeV}$. Not bad for a crude model!!

5. $100\,\mathrm{lb} = 45.3\,\mathrm{kg}$, $\rho = 19\,\frac{g}{cm^3} = 19000\,\frac{kg}{m^3}$

(a) $W_{crit} = \rho V_{crit} = \rho\,\frac{4}{3}\pi R_{crit}^3$

$$\Rightarrow\ R_{crit} = \left(\frac{W_{crit}}{\rho\,\frac{4}{3}\pi}\right)^{1/3} = \left(\frac{45.3\,\mathrm{kg}}{19000\,\frac{kg}{m^3}\cdot\frac{4}{3}\pi}\right)^{1/3} = 0.083\,\mathrm{m}$$

(b) $$E \sim 0.2M\left(\frac{R_{core}^2}{t^2}\right)\left(\sqrt{\frac{R_{core}}{R_{crit}}} - 1\right)$$

$$= 0.2\ 45.3\,\mathrm{kg}\ \frac{(1.2\cdot 0.083\,\mathrm{m})^2}{(10\,\mathrm{ns})^2}\ \overbrace{\left(\sqrt{1.2} - 1\right)}^{0.095}$$

$$= 0.2\cdot 45.3\cdot(1.2\cdot 0.083)^2\cdot 10^{16}\cdot 0.095\ \frac{kg\,m^2}{s^2} = 0.0085\times 10^{16}\,\mathrm{J}$$

$$= 8.5\times 10^{13}\ \mathrm{J}$$

(c) Convert to kt of conventional explosives using $1\,\mathrm{kt} = 4.2\times 10^{12}\,\mathrm{J}$

$$8.5\times 10^{13}\,\mathrm{J}\cdot\frac{1\,\mathrm{kt}}{4.2\times 10^{12}\,\mathrm{J}} = 20\,\mathrm{kt}$$

Really, I ought to have multiplied M by 1.2^3, because M is the mass of the core, not the critical mass, so that makes the explosion ~ 35kt