Photon-Proton Collision

We are going to use energy conservation and momentum conservation to show that a photon must have 54 MeV of energy to knock a proton at rest (in paraffin) out with 5.7 MeV of kinetic energy. Because only 14.6+5.3 ≈ 20 MeV is available, whatever the mystery radiation is in Fig. 2.23, it cannot be a photon (or γ-ray if you prefer that terminology).

What is special about the photon, and what is going to rule it out, is that the photon is massless.

In 1905, Einstein had published the theory of Special Relativity. We usually see $E=mc^2$ as being the most famous consequence, but that is only for a particle at rest. The more general statement is

$$E=\sqrt{p^2c^2+m^2c^4}$$

p is the momentum

For a particle at rest, the momentum is zero, and if you put $p=0$ in to

$$E=\sqrt{p^2c^2+m^2c^4}$$

you get back $E=mc^2$, which says that the mass of a particle at rest can be converted to energy.

For collision we are studying, we need the full equation, and we are going to use it four times!

First let's apply it to the proton, sitting still, waiting to be hit. We will use the subscript i, for initial, and f, for final. Initially the proton has $p_{p,i}=0$

momentum of proton, initially

So

$$E_{p,i}=m_pc^2$$

After the collision, the proton satisfies

$$E_{p,f}=\sqrt{p_{pf}^2c^2+m_p^2c^4}$$

$$=\underbrace{\sqrt{p_{pf}^2c^2+m_p^2c^4}-m_p^2c^2}_{}+m_p^2c^2$$

nobody can fault me for subtracting and adding the proton's rest energy if I please

This combination is called $K_{p,f}$ and it is what Chadwick measured to be as high as 5.7 MeV

Now we are going to work on the photon's contributions to energy and momentum conservation.

A photon also obeys $E=\sqrt{p^2c^2+m^2c^4}$ but a photon is massless. So this simplifies to $E=pc$ (almost!)

The photon is initially going to the right in figure 2-23. To give the greatest kick to the proton it needs to bounce back (to the left). Because we have photons going both right and left, we need to be more careful when we simplify

$$\sqrt{p^2c^2}$$

It actually simplifies to $|pc|$.

So we have

$$E_{\gamma,i} = |p_{\gamma,i}c| \text{ and } E_{\gamma,f} = |p_{\gamma,f}c|$$

and the absolute value signs say that a photon has positive energy whether or not its momentum is positive or negative. It is pretty standard to make the positive direction to the right, and so $p_{\gamma,i} > 0$. Meanwhile, we have argued that for the most kick $p_{\gamma,f}$ is to the left, so $p_{\gamma,f} < 0$.

Now use those facts to simplify:

$$E_{\gamma,i} = |p_{\gamma,i}c| = p_{\gamma,i}c \leftarrow \text{because } p_{\gamma,i} > 0$$

$$E_{\gamma,f} = |p_{\gamma,f}c| = -p_{\gamma,f}c \leftarrow \text{because } p_{\gamma,f} < 0$$

Or, $p_{\gamma,i} = E_{\gamma,i}/c$ and $p_{\gamma,f} = -E_{\gamma,f}/c$

We have one more bit of work to do before we can start using energy and momentum conservation.

Go back to $K_{p,f}$.

$$K_{p,f} = \sqrt{p_{p,f}^2c^2 + m_p^2c^4} - m_pc^2$$

$$= m_pc^2\left(\sqrt{1 + \frac{p_{p,f}^2c^2}{m_p^2c^4}} - 1\right)$$

In class, I claimed that if x is much less than 1, that

$$\sqrt{1+x} - 1 \approx \frac{1}{2}x$$

We could take the detour to further motivate the claim, but you can test it with a calculator. Even for x as large as 0.3 it works pretty well. The left-hand side is 0.14 and the right-hand side is 0.15.

So if $p_{p,f}c \ll m_pc^2$ (which amount to assuming $v_{p,f} \ll c$) then

the final velocity of the proton

the speed of light

$$K_{p,f} \approx m_pc^2 \cdot \frac{1}{2}\frac{p_{p,f}^2c^2}{m_p^2c^4} = \frac{1}{2}\frac{p_{p,f}^2}{m_p}$$

We will need to rearrange this equation to use it:

$$p_{p,f} = \sqrt{2mK_{p,f}}$$

Home stretch. Momentum conservation says:

$$\underset{E_{\gamma,i}/c}{p_{\gamma,i}} + 0 = \underset{-E_{\gamma,f}/c}{p_{\gamma,f}} + \underset{\sqrt{2m_p K_{p,f}}}{p_{p,f}}$$

Or, $$E_{\gamma,i} = -E_{\gamma,f} + \sqrt{2m_p c^2 K_{p,f}} \qquad (*)$$

Meanwhile, energy conservation says

$$E_{\gamma,i} + \cancel{m_p c^2} = E_{\gamma,f} + \cancel{m_p c^2} + K_{p,f}$$

Or, $$E_{\gamma,i} = -E_{\gamma,f} + K_{p,f} \qquad (**)$$

Add the two equations (*) and (**), and divide by 2, and get:

$$E_{\gamma,i} = \frac{K_{p,f} + \sqrt{2m_p c^2 K_{p,f}}}{2}$$

This is Reed's equation (2.34).

It may have taken a bunch of work to get it, but all the concepts are fundamentally important physics: $E=pc$ for a photon, $E=mc^2+K$ and $K=\frac{p^2}{2m}$ for a non-relativistic proton, plus the even more fundamental ideas of momentum conservation and energy conservation.

Anyway, we have the equation and now we stick in

$$K_{p,f} = 5.7\text{ MeV}$$
$$m_p c^2 = 938\text{ MeV}$$

and we get

$$E_{\gamma,i} = 57.4\text{ MeV}$$

Impossible! Only 20 MeV was available to make that photon.

In the next problem set, we'll see how supposing the existence of the neutron makes the reaction (2.33) plausible except now it is:

$${}^{4}_{2}He + {}^{9}_{4}Be \rightarrow {}^{12}_{6}C + \underset{\text{neutron instead of photon}}{{}^{1}_{0}n}$$