Mathematical Analysis: The Foundation of Calculus

Preliminary Syllabus

Latest version at brianhill.github.io/mathematical-analysis

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Course Short Name: **Mathematical Analysis**

Academic Year 2024-2025, Spring Semester (Terms 4 & 5)

Prerequisites

A semester of AP math quite helpful *Accessible* <===|===|+==> *Hard*

AP math will be quite helpful, but it is definitely not required. The more abstract thing that is required is a desire to think as precisely as a professional mathematician.

Materials

- *Calculus, 3rd Edition*, Michael Spivak, Publish or Perish, 2006. The Mathematical Association of America review begins, "This is the best Calculus textbook ever written." Need more be said? Anyway, this is the principal text for the course, but we will be using it mostly for its Parts I and II (its first eight chapters). This book's Part III can stand alone as the basis of a standard course in derivative and integral calculus. However our topic is the foundation of calculus, not calculus itself, and we will only get to Chapter 9 in Part III, which introduces the derivative.
- *What is Mathematics? 2nd Edition*, Richard Courant and Herbert Robbins, Oxford University Press, 1996. The review in the American Mathematical Monthly reiterates glowing commentary. "*What is Mathematics?* is one of the great classics, a sparkling collection of mathematical gems." "It should be in the hands of everyone, professional or otherwise, who is interested in scientific thinking." Its treatment and choice of topics are strongly influenced by the field of mathematical physics, and the book has been endorsed by Albert Einstein and Herman Weyl. We will use it for Part IV of the unit outline below.

Context and Overview

There is a wealth of material that logically fits in between standard high school mathematics and calculus. This material has been deemed too hard by most schools — both those with college prep tracks and with colleges themselves. It is simply skipped!

Only those that continue on to the junior level as mathematics majors get to see it, and they see it in a course that is often titled "Introduction to Analysis" or "Introduction to Mathematical Reasoning." It is perfect for the Deep Springs curriculum insofar as it requires no prerequisites beyond very good high school mathematics, and yet it is both more advanced and provocative than what is generally found in lower-division math courses.

The material is called "mathematical analysis" or "real analysis" and it includes many theorems that are absolutely essential to calculus but are typically used as if they are obvious and without proof.[1] What students usually move on to in their lower division courses, with the time freed up by skipping the foundations of calculus are all sorts of pragmatic results — like the chain rule as a method for simplifying derivatives, or integration by parts as a common trick for doing integrals, and even more mundane rules.

One might think that with only so much time to either obtain and apply practical results or to do mathematical proofs, that one should spend one's time on pedestrian problems. Michael Spivak, renowned for textbooks in mathematics ranging from the introductory level to textbooks used by mathematics graduate students, disagrees: "calculus ought to be the place in which to expect, rather than avoid, the strengthening of insight with logic." He goes on: "precision and rigor are neither deterrents to intuition, nor ends in themselves, but the natural medium in which to formulate and think about mathematical questions."

A course in the topics of mathematical analysis that undergird calculus prepares one to think rigorously, to understand real analysis the way mathematicians do, and to understand fundamental theorems that are beautiful and significant in their own right. As a treat we will even go beyond the confines of the real numbers and in our final unit introduce imaginary numbers (also known as complex numbers). This will allow us to conclude with the 1799 proof by Gauss of The Fundamental Theorem of Algebra.

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[1] As an example, The Intermediate Value Theorem — which says that if a function is continuous and it takes on the values *a* and *b*, then it must take on all the values between *a* and *b* — seems sufficiently obvious that it can be accepted without proof. However, the proof is well within a Deep Springer's reach, even someone who hasn't taken calculus and may decide not to go further in mathematics. The benefit of studying the proof of this particular theorem is that it forces one to be precise about the definition of continuity and about the properties of the real numbers. It allows one to enter into the mindset of the mathematician, whose main aim could be described as constructing logically rigorous and intellectually satisfying proofs starting from systems of definitions and axioms.

Unit Outline

I. Numbers
Integers
Rational Numbers
The Square Root of 2 is Irrational
Real Numbers

II. Functions and Limits
Functions as Rules for Mapping Real Numbers
Functions as a Set of Ordered Pairs
Depiction of Functions as Graphs
Limits

III. Properties of Continuous Functions
Continuity
The Intermediate Value Theorem
The Extreme Value Theorem
Least Upper Bounds
The Derivative

IV. Complex Numbers (alternatively, and depending on time and interest, I could scrap complex numbers and go a little further into derivatives and integrals)
The Square Root of -1
Properties of Complex Numbers
The Fundamental Theorem of Algebra

Studying Mathematics Texts

The number of pages you will be reading for each class is quite small: almost always less than ten. However, it is dense reading. To appreciate the way the arguments build, you need to understand both the logic and the mechanical steps for virtually every assertion in a mathematical text.

Math and science authors generally skip a few steps here and there. This keeps the size of a monograph manageable and keeps gives readers something to stay alert with. Whenever they skip steps, and you aren't sure you know exactly how to fill in the skipped steps in your head, put the book to the side, grab your pad, and figure out the skipped steps. On the other hand if you get thoroughly stumped at a step, note it as something you want to bring up in class, accept the result, and keep momentum by moving on.

When an author cross-references (to equations, to figures, to results in prior chapters, or to end-of-chapter problems), take time to follow the cross-reference. When an author suggests a little exercise for the reader, as in "we have used a*0 =0 already in a proof on page 6 (can you find where?)," stop and find where.

All of these things will help you put a solid foundation under what will become a weighty, mighty, and beautiful edifice.