Mathematical Analysis Exam 1

Feb. 10, 2025. If you get bogged down on a problem, move on and come back later. No part of any problem is meant to be long.

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1. Proofs of Things We All Know

Citing P10-P12 as you use them.

P is the set of positive numbers. The ***definition*** of the inequality $a < b$ is that $b - a$ is in P.

We wish to prove that if $0 < a < b$ and $0 < c < d$, then $ac < bd$ using the definition of inequality and whichever of P10-P12 you need to cite. You don't need to cite P1-P9 as you use them. I'll take you through the steps:

(a) What does $a < b$ say using the definition of inequality? And what does $c < d$ say?

(b) Prove that $ac < bc$. First, what does $ac < bc$ say using the definition of inequality?

(c) Prove that $bc < bd$. First, what does $bc < bd$ say using the definition of inequality?

(d) You now have $ac < bc$ and $bc < bd$. You are done because that implies $ac < bd$. But let's do a corollary.... Using what you have proved, show that if $0 < a < b$ then $a^2 < b^2$.

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2. More Proofs of Things We All Know

Citing P10-P12 as you use them and one more of the properties.

Suppose you know that $0 < a$ and $0 < b$, but unlike in 1(b), you don't yet know that $a < b$. However you do know that $a^3 < b^3$. We would like to prove that $a < b$! I am going to break this waaaaay down for you:

(i) Show that $b^3 - a^3 = (b - a)\left(a^2 + a\,b + b^2\right)$ just by doing a bit of multiplication to get six terms and then doing a bit canceling. You don't need to cite P1-P12.

(ii) Just using the definition of inequality given at the beginning of Problem 1 what does $a^3 < b^3$ mean?

(iii) Oops, there was no part (iii).

(iv) $b^3 - a^3 \in P$ and $b^3 - a^3 = (b - a)\left(a^2 + a\,b + b^2\right)$ tells us that $(b - a)\left(a^2 + a\,b + b^2\right) \in P$. What properties P1-P12 tell you that $a^2 + a\,b + b^2 \in P$?

(v) What property P1-P12 tells you that $\left(a^2 + a\,b + b^2\right)^{-1}$ exists?

(vi) I'll just tell you that if $c \in P$ then $c^{-1} \in P$ (it is super-easy to prove, but this problem is already getting long). So you can assume that $\left(a^2 + a\,b + b^2\right)^{-1} \in P$. How do you use that fact and (iv) and (v) to show that $b - a \in P$?

(vii) Going back to the definition of inequality, what is the familiar way to write write $b - a \in P$?

The properties P1–P9 have descriptive names which are not essential to remember, but which are often convenient for reference. We will take this opportunity to list properties P1–P9 together and indicate the names by which they are commonly designated.

(P1)	(Associative law for addition)	$a + (b + c) = (a + b) + c.$
(P2)	(Existence of an additive identity)	$a + 0 = 0 + a = a.$
(P3)	(Existence of additive inverses)	$a + (-a) = (-a) + a = 0.$
(P4)	(Commutative law for addition)	$a + b = b + a.$
(P5)	(Associative law for multiplication)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
(P6)	(Existence of a multiplicative identity)	$a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
(P7)	(Existence of multiplicative inverses)	$a \cdot a^{-1} = a^{-1} \cdot a = 1,$ for $a \neq 0.$
(P8)	(Commutative law for multiplication)	$a \cdot b = b \cdot a.$
(P9)	(Distributive law)	$a \cdot (b + c) = a \cdot b + a \cdot c.$

The three basic properties of numbers which remain to be listed are concerned with inequalities. Although inequalities occur rarely in elementary mathematics, they play a prominent role in calculus. The two notions of inequality, $a < b$ (a is less than b) and $a > b$ (a is greater than b), are intimately related: $a < b$ means the same as $b > a$ (thus $1 < 3$ and $3 > 1$ are merely two ways of writing the same assertion). The numbers a satisfying $a > 0$ are called **positive**, while those numbers a satisfying $a < 0$ are called **negative**. While positivity can thus be defined in terms of $<$, it is possible to reverse the procedure: $a < b$ can be defined to mean that $b - a$ is positive. In fact, it is convenient to consider the collection of all positive numbers, denoted by P, as the basic concept, and state all properties in terms of P:

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

- (i) $a = 0$,
- (ii) a is in the collection P,
- (iii) $-a$ is in the collection P.

(P11) (Closure under addition) If a and b are in P, then $a + b$ is in P.

(P12) (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

3. Inequalities

We are done with P1-P12. From here on, do not cite P1-P12. Just do the algebra.

(z) A warmup. If $|x| \le 1$ and $x_0 = 2$, what is the ***largest number*** that $|x + 2 x_0|$ could be? I'm just looking for an answer, not a proof.
(a) If $|x| \le 1$ but now you don't know what x_0 is, what is the ***largest number*** that $|x + 2 x_0|$ could be? HINT: Your answer will involve $|x_0|$ because I want a formula that works even if $x_0 < 0$.
(b) Using (a) to guide your intuition, if $|x| < 1$, what is the strongest ***inequality*** you can write for $|x + 2 x_0|$? HINT: All I have changed from (a) is ≤ to <, so you won't need to change much, but now you have to write down an inequality.
(c) Factor $(x + x_0)^2 - x_0^2$. HINT: It is in the form $a^2 - b^2$, which we incessantly factor into $(a + b)(a - b)$, but if if you prefer, you could first simplify it to $x^2 + 2 x_0 x$ and then factor it.
(d) Assuming the inequality you found in (b) holds. Apply that inequality to the factorization you found in (c), you now have an inequality for $\left|(x + x_0)^2 - x_0^2\right|$?
(e) If in addition to being given that $|x|<1$ you also know that $|x| < \frac{\epsilon}{2|x_0|+1}$ (where ϵ is some small and positive but otherwise unknown number), apply this new fact to the inequality you discovered in (d)?

COMMENT: This is the kind of thing we will be doing a lot of in Chapter 5, and Spivak has thoroughly prepared you for it with Chapters 1-4.

4. Induction and the Endowment

Let A_0 be an endowment. After one month, you get a return, r proportional to the endowment that tells you the new value of the endowment. $A_1 = A(1 + r)$.

However, running a college costs money, so you also draw off a fixed amount of money , a, from the endowment, and after one month you are left with

$A_1 = A_0(1 + r) - a.$

You repeat this process the next month, so

$A_2 = A_1(1 + r) - a$

etc., etc.

So at month n you have the **recursive formula** for A_n in terms of A_{n-1}.

$A_n = A_{n-1}(1+r) - a$

Someone tells you a **dubious solution** to the question of how much the endowment will be worth after n months, which you want to prove using induction (or disprove), and they say the solution is:

$A_n = A_0(1+r)^n - a\,\frac{1-r^n}{1-r}$ <== BTW, the "dubious" solution is wrong. Actually, $A_n = A_0(1+r)^n - a\,\frac{(1+r)^n-1}{r}$:)

(a) Is it true for $n = 1$? Check it. If yes, you are on the induction ladder.
(b) Make the **assumption** that the solution is true for $n = k$. Write down what assuming the **dubious solution** with $n = k$ says.
(c) You wish you could get to the next rung, A_{k+1}. Write down the **dubious solution** with $n = k + 1$.
(d) At least (c) tells you what you are trying to prove. So let's prove it. First write down the **recursive formula** (which is not at all dubious) with $n = k + 1$.
(e) The recursive formula has A_k in it, so use the **assumption** you wrote down in (b) to substitute into the **recursive formula** you wrote down in (d). Don't simplify yet.
(f) Now simplify (e). When you have simplified (e) answer whether this the same as what you wanted to prove in (c). If yes, you have proven that $n = k + 1$ is true assuming $n = k$ is true, which is showing we can get from one rung to the next on the ladder and the proof by induction is done. If no, the dubious solution is apparently false.

5. Composition of Functions

(a) Consider the hyperbola $y^2 - x^2 = 1$. Solve for y and only take the positive square root.
(b) Defining $f(x)$ to be what you found in (a), defining S to be the squaring function, s to be the square root function, and 1 to be the function that is always 1, write down a composition of functions expression for f. NOTE: x will not appear in your answer. We are looking for a composition of functions expression that says something more pure. EXAMPLE: If you had found $f(x) = x^2 - 1$ in (a), your answer for (b) would be $f = S - 1$.

6. Graphing, Domains

On the next page is a table of 15 values of $\cot\theta$, rounded to the nearest 0.1 and graph paper to use.

(a) Use those 15 values to plot the function $r = \cot\theta$ on the graph paper. *When you get to* $\frac{9\pi}{16}$ ***be sure to do what Spivak said to do on polar plots when you have negative values of r.***
(b) Freehand a line through your results.
(c) I only had you do the domain $\theta \in (0, \pi)$. In polar coordinates, the domain is typically $\theta \in [0, 2\pi)$. What values are definitely illegal and therefore excluded from $[0, 2\pi)$ for the function $r = \cot\theta = \frac{\cos\theta}{\sin\theta}$?

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Table[{θ, Round[Cot[θ], 0.1]}, {θ, Pi / 16, 15 Pi / 16, Pi / 16}]
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$$\left\{\left\{\frac{\pi}{16}, 5.\right\}, \left\{\frac{\pi}{8}, 2.4\right\}, \left\{\frac{3\pi}{16}, 1.5\right\}, \left\{\frac{\pi}{4}, 1.\right\}, \left\{\frac{5\pi}{16}, 0.7\right\}, \left\{\frac{3\pi}{8}, 0.4\right\}, \left\{\frac{7\pi}{16}, 0.2\right\}, \left\{\frac{\pi}{2}, 0.\right\}, \left\{\frac{9\pi}{16}, -0.2\right\}, \left\{\frac{5\pi}{8}, -0.4\right\}, \left\{\frac{11\pi}{16}, -0.7\right\}, \left\{\frac{3\pi}{4}, -1.\right\}, \left\{\frac{13\pi}{16}, -1.5\right\}, \left\{\frac{7\pi}{8}, -2.4\right\}, \left\{\frac{15\pi}{16}, -5.\right\}\right\}$$

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PolarPlot[5, {θ, 0, 2 Pi}, PolarGridLines → {Range[0, 2 Pi, Pi / 16], Range[5]}]
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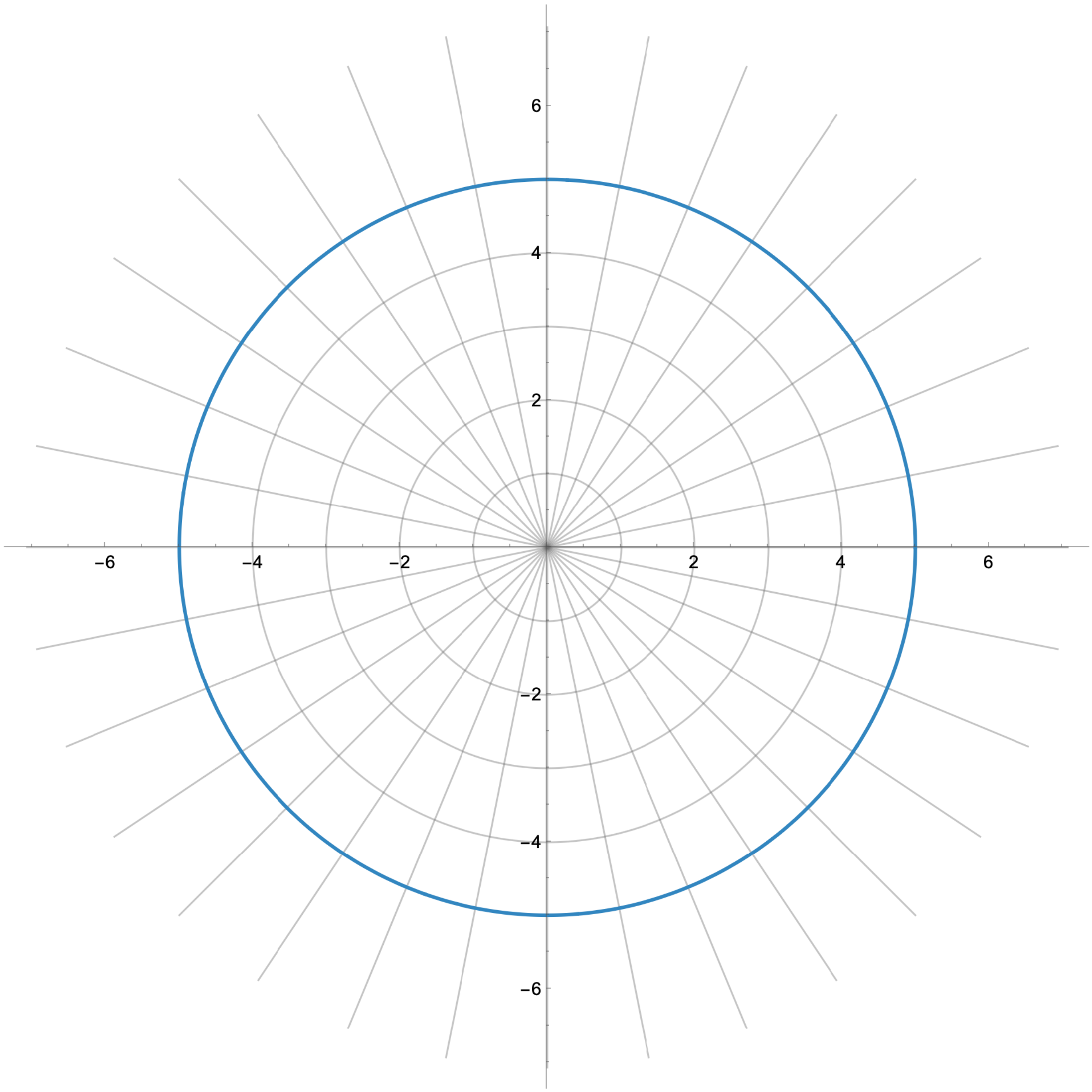

NAME: ______________________________

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2	/5
3	/5
4	/5
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6	/3
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TOTAL	/ 25