Mathematical Analysis Exam 2

Mar. 31, 2025. If you get bogged down on a problem, move on and come back later. There are 7 problems.

There are some algebraic tricks that we have used so many times, that they should be standard. If you can't remember the trick, that's another reason to move on.

Chapter 5 — Limits

Limits are the biggest topic, so there are three Chapter 5 problems.

1. Limits (4 pts)

Use the definition of the limit — a δ–ϵ proof starting with the limits poem — to show that:

$$\lim_{x \to 1} x^{1/3} = 1$$

HINT: After writing down what you are trying to prove, multiply $x^{1/3} - 1$ by $\frac{x^{2/3}+x^{1/3}+1}{x^{2/3}+x^{1/3}+1}$ and simplify.

2. Another Limits Problem (3 pts)

(a) What is the domain of:

$$f(x) = \frac{x^4-1}{x-1}$$

(b) What is

$$\lim_{x \to 1} f(x)$$

(Don't give a proof, just figure it out, but of course show your work on how you figured it out.)

3. Yet another Limits Problem (3 pts)

In Chapter 5, one of the whacky functions that Spivak asks us to contemplate is:

$$f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$$

(a) At what point(s) does this function have a limit and what is the limit?

(b) At whatever point(s) you claimed in (a) prove, starting from the limits poem, prove that the function has a limit.

Chapter 6 — Continuous Functions

4. Limit of the Inverse (5 pts)

Without using any lemmas (just starting from the definition of continuity), prove Theorem 1 Part (3) from Chapter 6. Here is the entire statement of Theorem 1:

THEOREM 1 If f and g are continuous at a, then

(1) $f + g$ is continuous at a,
(2) $f \cdot g$ is continuous at a.

Moreover, if $g(a) \neq 0$, then

(3) $1/g$ is continuous at a.

Again, I am only asking you to prove Part (3). But I want you to do it from scratch (just starting from the definition of continuity), and without quoting any lemmas.

DIRECTIONS: First clearly write down what you get to assume and what you need to show. Two points for just doing that right and nothing else.

Chapters 7 and 8 — Three Hard Theorems, Least Upper Bounds, And Uniform Continuity

5. Consequences of the Three Hard Theorems (2 pts)

The first of the three hard theorems announced in Chapter 7 was:

THEOREM 1 If g is continuous on $[a,b]$ and $g(a) < 0 < g(b)$, then there is some x in $[a,b]$ such that $g(x) = 0$.

Show using this theorem,

That $x^{179} - \sin^{13} x = 44$ has at least one solution.

HINT: What is the greatest and least value that $\sin^{13} x$ can possibly be?

6. Bounds of Functions (3 pts)

For Each of the Following Functions, Answer these six things: (a) Is the function bounded above (b) If it is bounded above, what is the least upper bound that you can give for the function; (c) If you have an answer to (b), is that value taken on by the functionl (d) Is the function bounded below; (e) If it is bounded below, what is the greatest lower bound that you can give for the function; (f) If you have an answer to (e), is that value taken on by the function?

(i) $f(x) = x^4$ on (-1,1)
(ii) $f(x) = x^4$ on $\mathbf{R}$
(iii) $f(x) = \frac{1}{\sin^2(x)}$ on $(0, \frac{\pi}{2}]$

DIRECTIONS: Your answer will have 18 parts: (i)(a), (i)(b), ... , (iii)(f). Organize your answer making a table with three rows and six columns. Put an X in any inapplicable table entries.

7. Uniform Continuity (5 pts)

Spivak gave this definition in the Appendix to Chapter 8:

DEFINITION The function f is **uniformly continuous on an interval** A if for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x and y in A,

$$\text{if } |x-y| < \delta, \text{ then } |f(x) - f(y)| < \varepsilon.$$

For the function $f(x) = \sqrt{|x|}$ on the interval $A = [-1,1]$, for any ϵ, find the δ that works on the entire interval, and thereby prove that with this δ, uniform continuity is satisfied.

HINT: There is a point where this function is the most nutzo. First find the δ that works only at that point. Then after you have done that, perhaps it will be easy to make a convincing argument that this δ or one closely related to it works everywhere on A.

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2	/3
3	/3
4	/5
5	/2
6	/3
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TOTAL	/ 25