# Mathematical Analysis Final

June 17, 2021. NONE OF THESE IS MEANT TO BE VERY LONG. If any problem seems very long, just move on to the next one.

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## 1. Limits (6 pts)

Use the definition of the limit — a $\delta$–$\epsilon$ proof starting with the limits poem — to show that:

$\lim_{x \to 1} x^{1/3} = 1$

HINT: After writing down what you are trying to prove, multiply $x^{1/3} - 1$ by $\frac{x^{2/3}+x^{1/3}+1}{x^{2/3}+x^{1/3}+1}$ and simplify.

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## 2. Bounds (6 pts)

For the following three functions on the domain (0,1] Answer each of: (i) does the function have a lower bound. If yes, (ii) what is the greatest lower bound, and (iii) does the function take on its greatest lower bound in this domain?

(a) $\frac{1}{x}$
(b) $\sin \frac{1}{x}$
(c) $-x^2$

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## 3. Consequences of the Three Hard Theorems (2 pts)

The first of the three hard theorems announced in Chapter 7 was:

**THEOREM 1** If $g$ is continuous on $[a, b]$ and $g(a) < 0 < g(b)$, then there is some $x$ in $[a, b]$ such that $g(x) = 0$.

Show using this theorem,

That $x^{179} - \sin^{13} x = 44$ has at least one solution.

# 4. Proofs Involving the Three Hard Theorems (3 pts)

Consider again the first of the three hard theorems:

**THEOREM 1** If $g$ is continuous on $[a, b]$ and $g(a) < 0 < g(b)$, then there is some $x$ in $[a, b]$ such that $g(x) = 0$.

This theorem begs to be generalized, and indeed, the first theorem proved in Chapter 7 was:

**THEOREM 4** If $f$ is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there is some $x$ in $[a, b]$ such that $f(x) = c$.

Prove Theorem 4 from Theorem 1. HINT: Draw $g$. Draw $f$. Do your drawings suggest that a $g(x)$ that is related to $f(x)$ to which you can apply Theorem 1, and thereby prove Theorem 4? *Your drawings aren't the proof.* Your drawings are just to help you realize what $g(x)$ you need to consider.

# 5. Derivatives (5 pts)

You have found (using the binomial theorem), that if $f(x) = x^n$, then $f'(x) = n x^{n-1}$. Surprisingly, even though you only proved this using the binomial theorem for integer $n$, it is true for all real $\alpha$: if $f(x) = x^\alpha$, then $f'(x) = \alpha x^{\alpha-1}$.

(i) Plug $\alpha = 1/3$ into this formula to get the derivative of $x^{1/3}$. (ii) Evaluate what you got in (i) for $x = 1$. (iii) Now that you have an answer, show that your answer to (ii) is correct starting from the definition of the derivative:

$$f'(1) \equiv \lim_{h \to 0} \frac{(1+h)^{1/3}-1}{h}$$

HINT: Very much like Problem 1, multiply through by $\frac{(1+h)^{2/3}+(1+h)^{1/3}+1}{(1+h)^{2/3}+(1+h)^{1/3}+1}$. Also, you can use any formulas for simplifying limits (like the limit of the sum is the sum of the limits).

# 6. Complex Numbers (3 pts)

Write $\sqrt{3 + 4i}$ in the form $a + bi$.

HINT: Start with $\sqrt{3 + 4i} = a + bi$ and square both sides. Then solve for $a$ and $b$. There are two solutions.