# **Uniform Continuity**

## For Chapter 8 Appendix, Problem 2, p. 144

- 2. (a) Prove that if f and g are uniformly continuous on A, then so is f + g.
  - (b) Prove that if f and g are uniformly continuous and bounded on A, then fg is uniformly continuous on A.
  - (c) Show that this conclusion does not hold if one of them isn't bounded.
  - (d) Suppose that f is uniformly continuous on A, that g is uniformly continuous on B, and that f(x) is in B for all x in A. Prove that  $g \circ f$  is uniformly continuous on A.

### 2(a) Sums

Prove that the sum of two uniformly continuous function is uniformly continuous.

Since f is uniformly continuous on A there is a  $\delta_f$  such that for all x and y in A,

 $|f(x) - f(y)| < \frac{\epsilon}{2},$ 

provided that

 $\|x-y\|<\delta_f.$ 

Similarly,

 $|g(x)-g(y)|<\frac{\epsilon}{2},$ 

provided that

 $|x - y| < \delta_g$ 

Now choose  $\delta$  to be the smaller of  $\delta_f$  and  $\delta_g$ . Then

 $|f(x) + g(x) - (f(y) + g(y))| \le |f(x) - f(y)| + |g(x) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ 

Voila.

## 2(b) Products

Prove that the product of two uniformly continuous function is uniformly continuous, provided that both functions are bounded on the interval in question.

As in 2(a), we will have a  $\delta_f$  and  $\delta_g$ . This time choose the  $\epsilon$  for f to be  $\frac{\epsilon}{2N}$ . E.g., our  $\delta_f$  will be sufficiently small such that

 $|f(x) - f(y)| < \frac{\epsilon}{2N}$ 

where *N* is the bound on *g*. E.g., |g(x)| < N.

Similarly, choose the  $\epsilon$  for g to be  $\frac{\epsilon}{2M}$ . E.g., our  $\delta_g$  will be sufficiently small such that

 $|g(x)-g(y)| < \frac{\epsilon}{2M}$ 

where *M* is the bound on *f*. E.g., |f(x)| < M.

Now we consider

$$\begin{split} |f(x)g(x) - f(y)g(y)| &= \\ |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| &\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| = \\ |f(x) - f(y)| |g(x)| + |f(y)| |g(x) - g(y)| &< \\ |f(x) - f(y)| N + M |g(x) - g(y)| &< \frac{\epsilon}{2N} N + M \frac{\epsilon}{2M} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{split}$$

Voila.

#### Counterexample asked for in 2(c)

The product of two functions that are uniformly continuous may not be uniformly continuous if one of them is unbounded. Of course the only way this can happen is if the interval is unbounded, because a continuous function on a bounded interval is bounded.

Example: let  $f(x) = \sin x$  and g(x) = x. Let the interval be  $[0, \infty)$ . Both f and g are uniformly continuous even though g is unbounded. That's because g's slope is constant.

Let's graph the product:



Let's consider  $x = 2\pi$  and  $x = 4\pi$ . Let's consider  $\epsilon = 1$ . At  $x = 2\pi$ , that means  $|x \sin x| < 1$ . Let's blow up that region of the plot:

```
In[*]:= Plot[xSin[x], \{x, 2Pi - 1, 2Pi + 1\}, PlotRange \rightarrow \{\{2Pi - 1, 2Pi + 1\}, \{-1, 1\}\}, AspectRatio \rightarrow 1]
Out[*]:= 10^{-1} \frac{10^{-1}}{0.5^{-1}} \frac{10^{-1}
```

Eyeballing the plot, it appears that a  $\delta$  of about 0.2 will do the job for  $\epsilon$ =1.

Now let's make a similar plot but at  $x = 4 \pi$ .

```
In[*]:= Plot[xSin[x], {x, 4Pi - 1, 4Pi + 1},
PlotRange → {{4Pi - 1, 4Pi + 1}, {-1, 1}}, AspectRatio → 1]
Out[*]=
0.0
0.5
0.5
0.0
12.0
12.5
13.0
13.5
```

It is clear that the  $\delta$  of about 0.2 will not do the job because the function is steeper here (twice as steep in fact).

You could do  $\delta$  of about 0.1, but then if you go out twice as far again, to  $x = 8 \pi$ , you'll need a  $\delta$  of about 0.05 because the function is twice again as steep:

```
In[*]:= Plot[xSin[x], \{x, 8 Pi - 1, 8 Pi + 1\}, PlotRange \rightarrow \{\{8 Pi - 1, 8 Pi + 1\}, \{-1, 1\}\}, AspectRatio \rightarrow 1]
Out[*]:= 1.0 \begin{bmatrix} 1.0 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5
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### 2(d) Composition

Here again is the statement of Part (d):

Suppose that f is uniformly continuous on A, that g is uniformly continuous on B, and that f(x) is in B for all x in A. Prove that  $g \circ f$  is uniformly continuous on A.

The proof of this is going to have to go a lot like the proof of Theorem 2 of Chapter 6 on p. 115, which starts out like this:

**THEOREM 2** If g is continuous at a, and f is continuous at g(a), then  $f \circ g$  is continuous at a. (Notice that f is required to be continuous at g(a), not at a.)

**PROOF** Let  $\varepsilon > 0$ . We wish to find a  $\delta > 0$  such that for all x,

$$\begin{split} \text{if } |x-a| < \delta, \text{ then } |(f \circ g)(x) - (f \circ g)(a)| < \varepsilon, \\ \text{i.e., } |f(g(x)) - f(g(a))| < \varepsilon. \end{split}$$

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Notice that the roles of *f* and *g* have been interchanged.

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f g

If you look at the rest of the proof of Theorem 2 of Chapter 6, you'll see that it indeed has almost the exactly the same outline as our proof that follows:

#### PROOF

Because g is uniformly continuous on B we know that for f(x) and f(y) (which by assumption are in B provided x and y are in A) that there is a  $\delta_g$  such that  $|g(f(x)) - g(f(y))| < \epsilon$  provided that  $|f(x) - f(y)| < \delta_g$ .

But we also know that since f is uniformly continuous on A that there is a  $\delta_f$  such that  $|f(x) - f(y)| < \delta_g$ provided that  $|x - y| < \delta_f$ . ( $\delta_g$  is just playing the role of the  $\epsilon$  in the definition of uniform continuity for f.)

To summarize: the  $\epsilon$  requires a certain  $\delta_g$ , and this  $\delta_g$  transitively requires a certain  $\delta_f$ . This  $\delta_f$  is the  $\delta$  we need to make

 $|g(f(x))-g(f(y))|<\epsilon$ 

for any x, y in A with  $|x - y| < \delta$ .

NOTHING NEW, BUT ANOTHER PROOF/RESTATEMENT

We want to show that given any  $\epsilon$ , there is a  $\delta$  that keeps  $|g(f(x)) - g(f(y))| < \epsilon$ .

So we START with this epsilon, and use the assumption that g is uniformly continuous. That tells us that there is a  $\delta_g$ . Now that we know what that  $\delta_g$  has to be, we use the assumption that f is uniformly continuous. That tells us there is a  $\delta_f$  that keeps  $|f(x) - f(y)| < \delta_g$ . This  $\delta_f$  is what we need to use as our  $\delta$ .