Math Analysis	Problem Set 1	2021-03-18

Problem 5 Chapter 1. Additionally Problem 18 (a)-(c) that we'll do in class

5(i) $a<b$ means $b-a>0$ (or $b-a$ is in P)
$c<d$ means $d-c>0$ (or $d-c$ is in P)
By (PII) (closure under addition of P)
$b-a+(d-c)>0 \Rightarrow b+d-(a+c)>0$ by standard rearrangements
$\Rightarrow a+c<b+d$

(ii) $a<b$ means $b-a$ is in P
$-b<-a$ is what we want to show
and it is equivalent to $-a-(-b)$ is in P
But standard rearrangements show that
$-a-(-b)=b-a$ which by assumption is in P.

(iii) $c>d$ means $d<c$. We also know from part (ii) that $d<c$ is equivalent to $-c<-d$. So this is from here on the same as part (i), except c is $-c$ and d is $-d$.

(iv) $a<b$ means $b-a$ is in P. Now P is closed under multiplication and c is in P, so $c\cdot(b-a)$ is in P, and from that you get $bc-ac$ is in P which means $bc<ac$.

(v) $a<b$ means $b-a>0$. $c<0 \Rightarrow -c>0$.
Closure under multiplication of P means
$(b-a)(-c)>0$. Rearrange and get
$ac-bc>0 \Rightarrow bc<ac$

(vi) If $a>1$ then $a-1$ is in P.
$a^2-a=a(a-1)$ is in P by closure of P under multiplication. So $a<a^2$.

5(vii) If $0<a<1$, we know that a is in P $a<1$. $a<1$ means $1-a$ is in P. By closure $a\cdot(1-a)=a-a^2$ is in $P \Rightarrow a^2<a$.

(viii) If either a or c is zero it is trivial. So we have to deal with the case where a and c are both nonzero.

$0<a<b$ and $0<c<d$

We are trying to show that $ac<bd$.

I.e., we are trying to show $bd-ac$ is in P.

$bd-ac = bd-bc+bc-ac = b\cdot(d-c)+(b-a)\cdot c$

These are four positive things so by closure under addition and multiplication we have that $bd-ac$ is in P.

5(ix) Let c be a and d be b.

By (viii) we had $ac<bd$ which becomes $a^2<b^2$

5(x) By trichotomy $a<b$, $a=b$ or $a>b$

Assume $a=b$. Then $a^2=b^2$ in violation of $a^2<b^2$.

Assume $a>b$. Then by (ix) backwards $b^2<a^2$ in violation again of $a^2<b^2$. So that leaves $a<b$ ← the only possibility of the trichotomy that we have not ruled out.

18 (a)-(c) not assigned, but I liked it

18(a) Plug $x_+ = \frac{-b+\sqrt{b^2-4c}}{2}$ into x^2+bx+c

$$\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b+\sqrt{b^2-4c}}{2}\right) + c$$

$$= \cancel{\frac{b^2}{4}} - \cancel{\frac{b\sqrt{b^2-4c}}{2}} + \frac{1}{4}(\cancel{b^2}-\cancel{4c}) - \cancel{\frac{b^2}{2}} + \cancel{b\frac{\sqrt{b^2-4c}}{2}} + \cancel{c}$$

$= 0$ ✓✓

Plug $x_- = \frac{-b-\sqrt{b^2-4c}}{2}$ into x^2+bx+c

$$\left(+\frac{b+\sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b-\sqrt{b^2-4c}}{2}\right) + c$$

$$= \cancel{\frac{b^2}{4}} + \frac{1}{2}\cancel{b\sqrt{b^2-4c}} + \frac{\cancel{b^2}-\cancel{4c}}{4} - \cancel{\frac{b^2}{2}} - \cancel{b\frac{\sqrt{b^2-4c}}{2}} + \cancel{c}$$

$= 0$ ✓

(b) $$x^2+bx+c = \underbrace{\underbrace{\left(x+\frac{b}{2}\right)^2}_{\geq 0 \text{ because it is a square}} \underbrace{-\frac{b^2}{4}+c}_{>0 \text{ by assumption}}}_{>0 \text{ so can never be zero}}$$

(c) Think of y as b and y^2 as c

$$b^2-4c = y^2-4y^2 = -3y^2 < 0 \quad \text{if } y \neq 0$$

So we can use (b) to say there is no solution. and x^2+xy+y^2 is always >0 (assuming $y \neq 0$).