Mathematical Analysis — PS 6 — Vectors

April 12, 2021

This problem set is designed to round out your knowledge of vectors, introduced by Spivak in Appendix 1 of Chapter 4.

Your next reading (for Monday, April 12th) is pp. 90-96 of Chapter 5. Chapter 5 is on limits. I wanted to do Appendix 3 of Chapter 4 next, but we need to start limits so that we can spend three classes on it. I think we will have time to come back to Appendix 3 of Chapter 4 in Term 6.

Rotations of Vectors and the Dot Product

In class, we did Problems 1(a), 1(c), 2(b), 2(c), and 3(a). Review those carefully. The big picture result of those problems are that we have formulas for:

- rotations of vectors
- the dot product of two vectors
- the norm or length of a vector
- and finally, we have shown that that dot product is invariant under rotations — that was Problem 3(a) that Amelia finished the algebra for at the board

In 1(a), 1(c), and 3(a), we were using θ to denote the amount of rotation. In 3(b) Spivak uses θ for the angle between two vectors. Be clear which you are using θ for in what follows if there is any ambiguity.

Your First Problem

Do Problem 3(b). Use everything that Spivak has built up to this point. It is pretty easy if you are crafty. In particular use invariance of the dot product under rotations.

Rotations of Vectors and the “Cross” Product

In problem 4, Spivak defines the determinant of two vectors:

$$\det(v,\ w) \equiv v_1\, w_2 - v_2\, w_1$$

Physicists call this the “cross” product. It is heavily used for angular momentum, magnetic fields, magnetic forces, and for turbulence in fluid flow.

Your Second Problem

Just as we did in 2(b) for the dot product, prove the basic properties of the cross product, which are:

$\det(v,\ w) = -\det(w,\ v)$
$\det(v,\ w + z) = \det(v,\ w) + \det(v,\ z)$
$a \cdot \det(v,\ w) = \det(a v,\ w) = \det(v,\ a\, w)$ (a is just a number, not a vector in this formula)

Your Third Problem

Do Problem 5(b). This problem shows that like the dot product, the cross product is also invariant under rotations.

An Example

Your Fourth Problem

Here are two vectors that make a 45° angle with each other:

$v = (3,\ 3)$
$w = (0,\ 4)$

(a) Calculate $v \cdot w$ using their components.
(b) Calculate $v \cdot w$ using the formula $v \cdot w = \| v \| \cdot \| w \| \cdot \cos\theta$
(c) Calculate $\det(v,\ w)$ using their components.
(d) Calculate $\det(v,\ w)$ using the formula $\det(v,\ w) = \| v \| \cdot \| w \| \cdot \sin\theta$

Proof that $\det(v,\ w) = \| v \| \cdot \| w \| \cdot \sin\theta$

In part (d) of the example, we used a formula we haven't proved yet.

Your Fifth Problem

Do Problem 7, which is to prove $\det(v,\ w) = \| v \| \cdot \| w \| \cdot \sin\theta$, using the same crafty methods you did to prove 3(b).