

## Newton — Problem Set 3 — The Next Six Lemmas

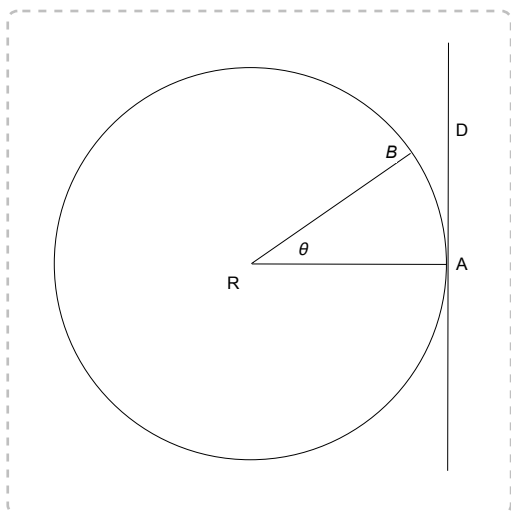
Due Tuesday, Sep. 27 (beginning of class). These are modern problems to illustrate, with specific figures that we can analyze with algebra, and specific values to plug in to the algebra, the meaning and validity of some of Newton's lemmas.

You will be working with small quantities. The only way it will be convincing that Newton is right about what is happening to these small quantities is to be quite careful both in keeping plenty of decimal places around when doing calculations, and in being accurate when making graphs.

### 1. Lemma 7 Example: Equality of Sine, Tangent, and Arc Length of a Circle, in the Limit

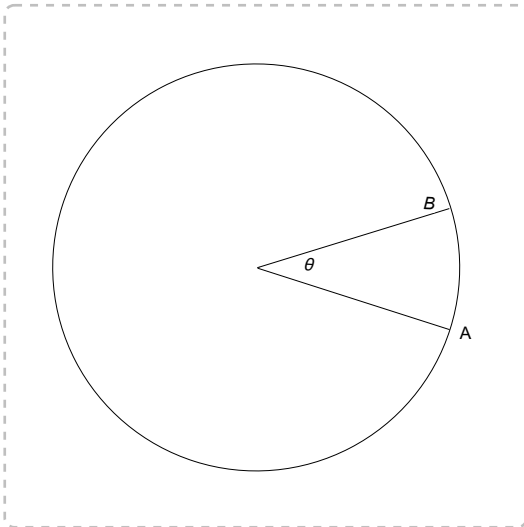
The circle has circumference  $2\pi r$ , and if you measure angles in radians, the arc length of a wedge of a circle is simply  $\theta r$ . There are by definition of the radian  $2\pi$  radians in a full circle. There are by definition of the degree  $360^\circ$  in a full circle. Because  $2\pi$  radians =  $360^\circ$ , you have a conversion factor between these two ways of measuring angles. These facts are fundamental. We are going to be working in radians. The attractiveness of the unfamiliar unit of radians is so that the arc length of a wedge of a circle is simply  $\theta r$ . If you work in degrees, then the arc length is the messier  $\theta r \frac{2\pi}{360^\circ}$ .

In the following picture of a circle, we have Newton's Lemma 7 labels applied to a circle. R is the center of the circle. A is a tangent to the circle. D is the place on the tangent where the radius RB would hit the tangent if it were extended. We also have  $\theta$ , the angle of ARB.



(a) Assume the circle has radius 1 for convenience. If  $\theta$  is measured in radians, what is the arc length AB? Using standard trig functions, what is the length AD?

(b) Using standard trig functions, what is the chord AB? This part is tricky!! Perhaps redrawing the diagram like this gives you a hint of how to get a formula for that chord length (whispers: bisect  $\theta$ ):



(c) You now have a formula for the chord, the arc length, and the tangent. Get a calculator, make sure it is in radians mode (because one of the three formulas requires radians), and compare these three quantities for 1 radian, 0.1 radian, and 0.01 radian. They should tend toward one another for small  $\theta$  if Newton is right.

*NB: Of course they all tend to zero as  $\theta$  goes to zero, that isn't what Newton was getting at, and it isn't significant to just notice they are all tending to zero. What is significant and convincing, and what your comparisons should show is that the ratios tend to unity.*

## 2. Arc Lengths and Chord Lengths of Parabolas

From  $(0, 0)$  to the point  $(a, a^2)$  on the parabola  $y = x^2$  the arc length is

$$\frac{1}{2} a \sqrt{1 + 4a^2} + \frac{1}{4} \ln(2a + \sqrt{1 + 4a^2}).$$

This is a little-known formula (at least I had never heard of it). I got it by doing a web search which yielded: [https://www.math.drexel.edu/~tolya/arc\\_length\\_x^2.pdf](https://www.math.drexel.edu/~tolya/arc_length_x^2.pdf). We are just going to use this parabolic arc length formula.

(a) What is the chord length from  $(0, 0)$  to the point  $(a, a^2)$ ? (This only requires the Pythagorean theorem.)

(b) Compare the chord length and the arc length for  $a = 3$ ,  $a = 1$ , and  $a = 1/3$ . The same note about what is significant and convincing following 1(c) applies here.

### 3. Lemma 8 applied to Circles

In Lemma 8, Newton is talking about areas. We can calculate these areas in the case of a circle, much as we calculated chord lengths in Problem 1.

(a) What is the area of the “triangle” bounded by RA, RB, and the arc AB? What is the area of the triangle bounded by RA, RD, and AD?

(b) What is the area of the triangle bounded by RA, RB, and the chord AB? This could get sticky.

(c) Do the same thing as you did in 1(c) for these three areas.

### 4. Lemma 10 — An Example

We know that for constant forces, the displacement caused by the force goes as the square of the elapsed time. If you want to use modern terms, for a constant force and a constant acceleration,  $a = F/m$ , and  $d = \frac{1}{2} a t^2$ .

Newton says that it matters not that the force is constant!! He says the displacement still goes as  $t^2$ . But only initially, meaning only in the limit of sufficiently small times.

Ok, so let's check on him with an example. Here is a variable force:  $F(t) = F_0 \cos \omega t$ . For a mass  $m$ , this results in an acceleration,  $a(t) = \frac{F_0}{m} \cos \omega t$ . I say that the resulting velocity is  $v(t) = \frac{F_0}{m\omega} \sin \omega t$ . For those whose calculus is fresh, you might want to quickly check that  $\frac{d}{dt} v(t)$  is indeed  $a(t)$ . Furthermore, I say that the resulting displacement is  $x(t) = \frac{F_0}{m\omega^2} (1 - \cos \omega t)$ . Again, if you have the calculus fresh enough, check that  $\frac{d}{dt} x(t)$  is indeed  $v(t)$ . Also, I arranged the constants so that  $v(0) = 0$  and  $x(0) = 0$ .

(a)  $\frac{F_0}{m\omega^2}$  must have dimensions of length. Let's choose the constants so that this combination is 1 meter. Meanwhile,  $\omega$  must have dimensions of inverse time. Let's choose it so that it is simply  $\frac{1 \text{radian}}{\text{second}}$ . Do a very nice graph of  $x(t) = 1 \text{ meter} \left(1 - \cos \frac{t \text{radian}}{1 \text{second}}\right)$  for the time interval -1 second to 1 second. You could use an online tool, or you could manually plot it.

(b) Superimpose onto this graph a graph of the function  $x(t) = \frac{1}{2} (1 \text{ meter}) \left(\frac{t}{1 \text{second}}\right)^2$ .

(c) Evaluate and compare  $x(t) = 1 \text{ meter} \left(1 - \cos \frac{t \text{radian}}{1 \text{second}}\right)$  and  $x(t) = \frac{1}{2} (1 \text{ meter}) \left(\frac{t}{1 \text{second}}\right)^2$  for  $t = 1$  second,  $t = 0.5$  seconds,  $t = 0.25$  seconds and  $t = 0.125$  seconds. As usual, since the quantities are tending toward zero, you will want to take ratios to observe something significant.