Newton — Problem Set 4 — Solution

1. A Very Strange Central Force Law

AREAS

(a) The rate at which the planet sweeps out area is proportional to:

$V_{left}(a-e)$ on the left

$V_{across}\, a$ on the bottom and top

and

$V_{right}(a+e)$ on the right

These rates must be equal by Proposition 1, so we learn

$$V_{across} = \frac{a-e}{a} V_{left} \quad \text{and} \quad V_{right} = \frac{a-e}{a+e} V_{left}$$

ANGLES

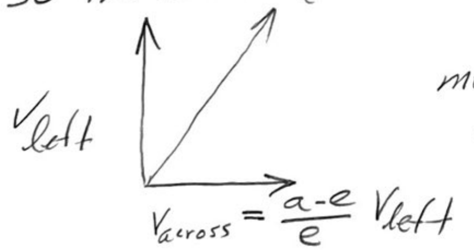
(b) The $\Delta QoM_{horizontal}$ is proportional to $V_{across} = \frac{a-e}{a} V_{left}$

quantity of motion

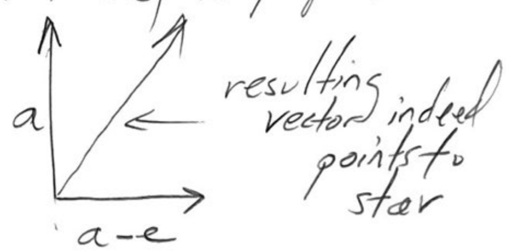
component in the horizontal direction at the lower-left corner

The $\Delta QoM_{vertical}$ is proportional to V_{left}

So the vector (which is the sum of these) is proportional to

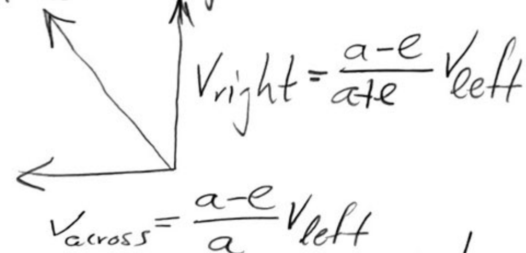


multiply by a and divide by V_{left}



MAGNITUDES

(c) At lower right



Length at lower left is:

$$\sqrt{\left(\frac{a-e}{a}\right)^2 + 1}\; V_{left}$$

Length at lower right is:

$$\sqrt{\left(\frac{a-e}{a}\right)^2 + \left(\frac{a-e}{a+e}\right)^2}\; V_{left}$$

Ratio is

$$\underbrace{\sqrt{\frac{1}{a^2} + \frac{1}{(a+e)^2}}}_{\text{lower right}} \Bigg/ \underbrace{\sqrt{\frac{1}{a^2} + \frac{1}{(a-e)^2}}}_{\text{lower left}}$$

WHAT A STRANGE FORCE LAW!

2. Radii of Curvature — Newton's Construction Involving Chords

(a) Below is a table showing the x-value, the slope of the chord leading to that x-value, and the place on the y-axis where the perpendicular to the chord intersects it.

In[7]:=
```
twoAPerpendicular[x_] := -x / (-Sqrt[1 - x^2] + 1)
```

In[12]:=
```
MatrixForm[
 Table[{x, twoAPerpendicular[x], -x twoAPerpendicular[x] + (-Sqrt[1 - x^2] + 1)},
  {x, DeleteCases[Range[-0.5, 0.5, 0.1], 0.0]}]]
```

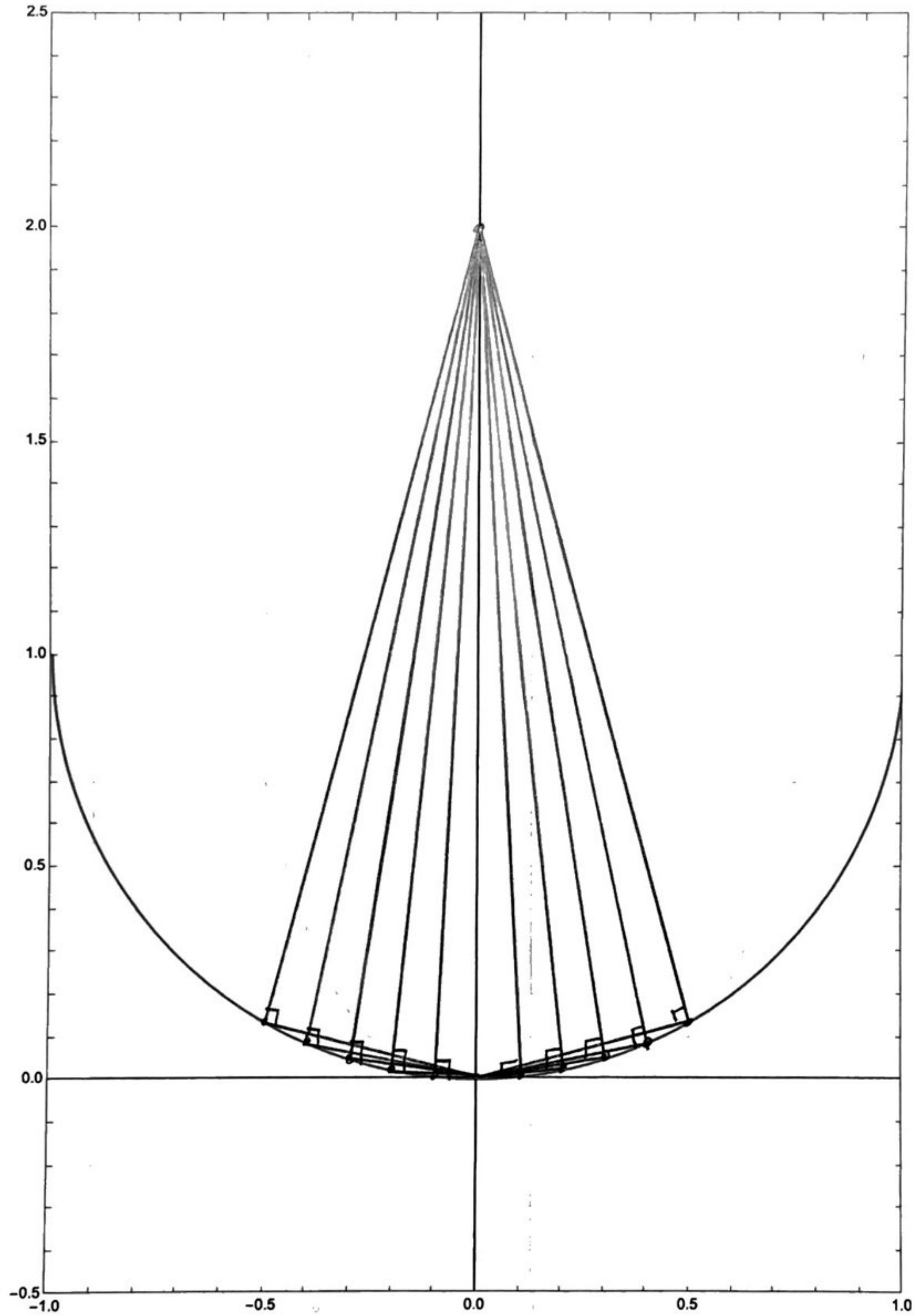
Out[12]//MatrixForm=

-0.5	3.73205	2.
-0.4	4.79129	2.
-0.3	6.51313	2.
-0.2	9.89898	2.
-0.1	19.9499	2.
0.1	-19.9499	2.
0.2	-9.89898	2.
0.3	-6.51313	2.
0.4	-4.79129	2.
0.5	-3.73205	2.

On the next page, I have plotted these results. Notice that for the circle, all the points go to exactly the same place. This is a property of the circle, not of other curves that are close to circular.

In[ ]:=
```
Plot[-Sqrt[1 - x^2] + 1, {x, -1, 1},
 PlotRange → {{-1, 1}, {-0.5, 2.5}}, AspectRatio → 3 / 2,
 GridLines → {Range[-1, 1, 0.1], Range[-0.5, 2.5, 0.1]}, Frame → True]
```

Out[ ]=



(b) Below is a table showing the x-value, the slope of the chord leading to that x-value, and the place on the y-axis where the perpendicular to the chord intersects it.

In[14]:=
```
twoBPerpendicular[x_] := -x / (5 x^4)
```

In[16]:=
```
MatrixForm[Table[{x, twoBPerpendicular[x], -x twoBPerpendicular[x] + (5 x^4)},
  {x, DeleteCases[Range[-1, 1, 0.1], 0.0]}]]
```

Out[16]//MatrixForm=

-1.	0.2	5.2
-0.9	0.274348	3.52741
-0.8	0.390625	2.3605
-0.7	0.58309	1.60866
-0.6	0.925926	1.20356
-0.5	1.6	1.1125
-0.4	3.125	1.378
-0.3	7.40741	2.26272
-0.2	25.	5.008
-0.1	200.	20.0005
0.1	-200.	20.0005
0.2	-25.	5.008
0.3	-7.40741	2.26272
0.4	-3.125	1.378
0.5	-1.6	1.1125
0.6	-0.925926	1.20356
0.7	-0.58309	1.60866
0.8	-0.390625	2.3605
0.9	-0.274348	3.52741
1.	-0.2	5.2

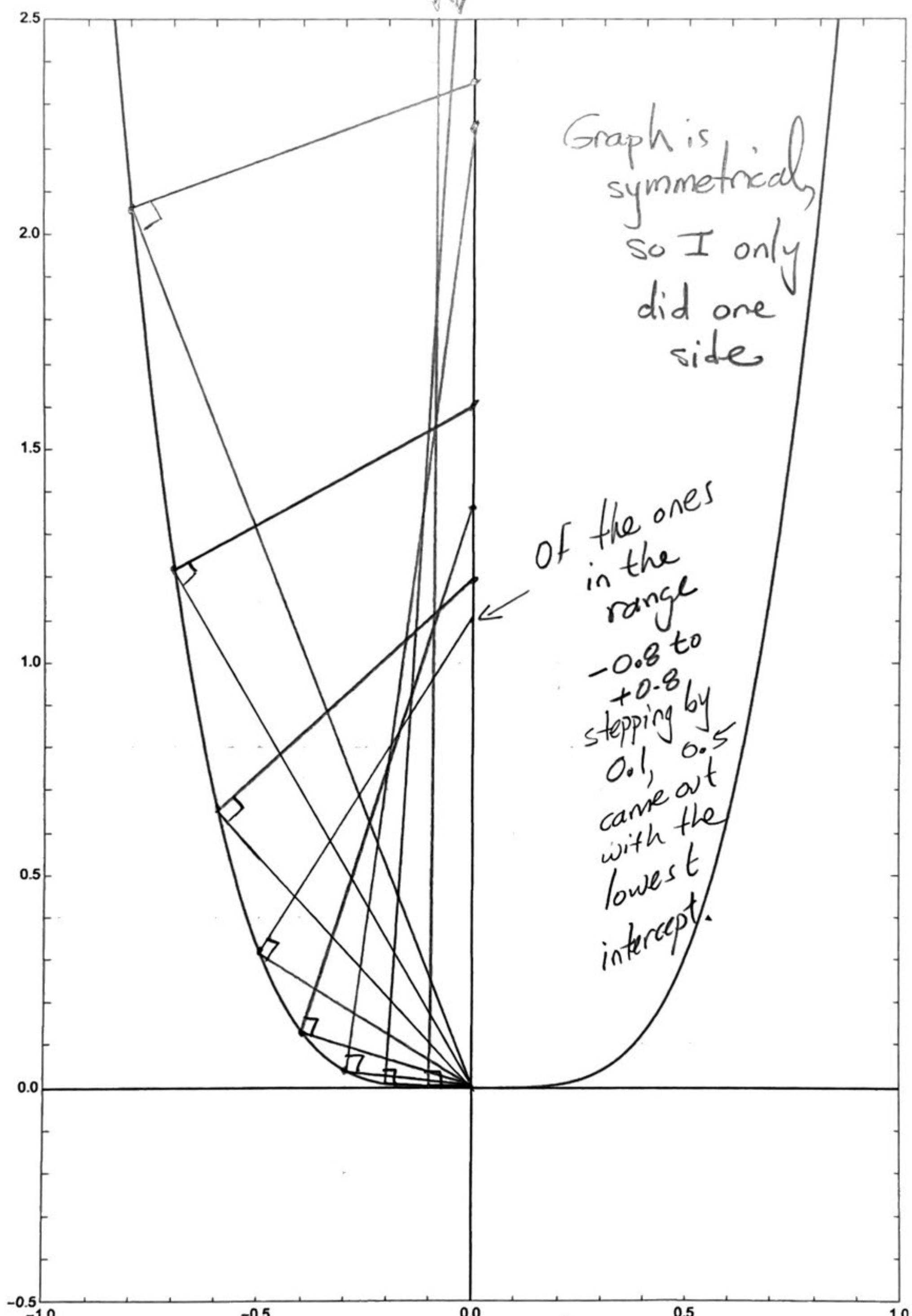
On the next page, I have plotted these results. Notice that for the quartic, there isn't convergence! The origin is a point on this curve with zero curvature.

intercept is at 20

intercept is at 5

In[ ]:=
```
Plot[5 x^4, {x, -1, 1}, PlotRange → {{-1, 1}, {-0.5, 2.5}}, AspectRatio → 3 / 2,
  GridLines → {Range[-1, 1, 0.1], Range[-0.5, 2.5, 0.1]}, Frame → True]
```

Out[ ]=



Bottom line: intercepts are not converging (at all)

(c) Below is a table showing the x-value, the slope of the chord leading to that x-value, and the place on the y-axis where the perpendicular to the chord intersects it.

In[23]:=
```
twoCPerpendicular[x_] := -x / (x^3 + x^2)
```

In[30]:=
```
MatrixForm[Table[{x, twoCPerpendicular[x], -x twoCPerpendicular[x] + (x^3 + x^2)},
  {x, DeleteCases[Range[-0.5, 1.0, 0.1], 0.0]}]]
```

Out[30]//MatrixForm=

-0.5	4.	2.125
-0.4	4.16667	1.76267
-0.3	4.7619	1.49157
-0.2	6.25	1.282
-0.1	11.1111	1.12011
0.1	-9.09091	0.920091
0.2	-4.16667	0.881333
0.3	-2.5641	0.886231
0.4	-1.78571	0.938286
0.5	-1.33333	1.04167
0.6	-1.04167	1.201
0.7	-0.840336	1.42124
0.8	-0.694444	1.70756
0.9	-0.584795	2.06532
1.	-0.5	2.5

On the next page, I have plotted these results. This is the more typical case that Newton is expecting. Unlike for the circle they don't converge identically. Unlike the quartic at $x = 0$ though, they do converge.

Although I haven't plotted the additional ten points below on the next page, they help see numerically that they are converging to 1.0.

In[34]:=
```
MatrixForm[Table[{x, twoCPerpendicular[x], -x twoCPerpendicular[x] + (x^3 + x^2)},
  {x, Range[-0.09, 0.09, 0.02]}]]
```

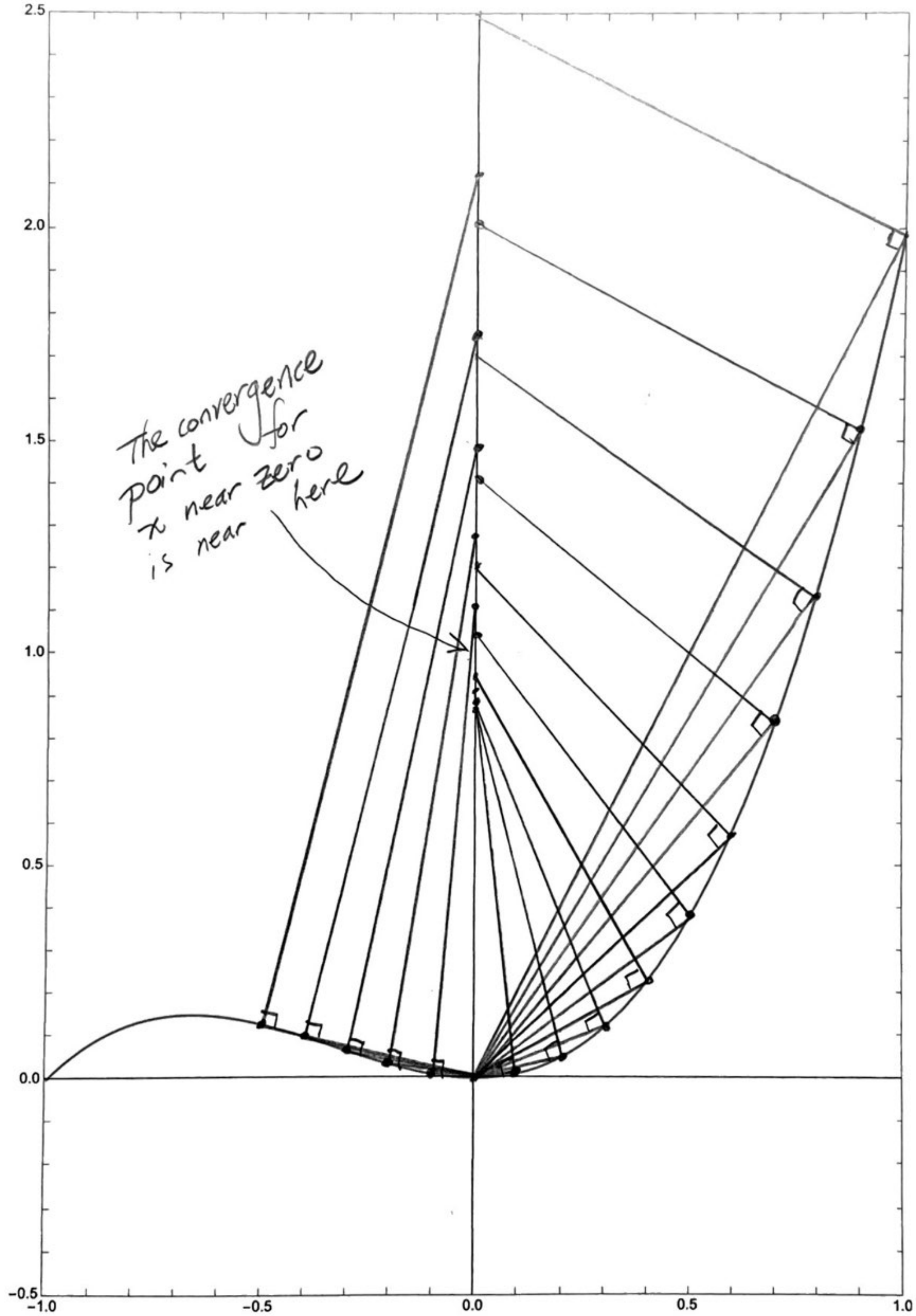
Out[34]//MatrixForm=

-0.09	12.21	1.10627
-0.07	15.361	1.07983
-0.05	21.0526	1.05501
-0.03	34.3643	1.0318
-0.01	101.01	1.0102
0.01	-99.0099	0.9902
0.03	-32.3625	0.971801
0.05	-19.0476	0.955006
0.07	-13.3511	0.939822
0.09	-10.1937	0.92626

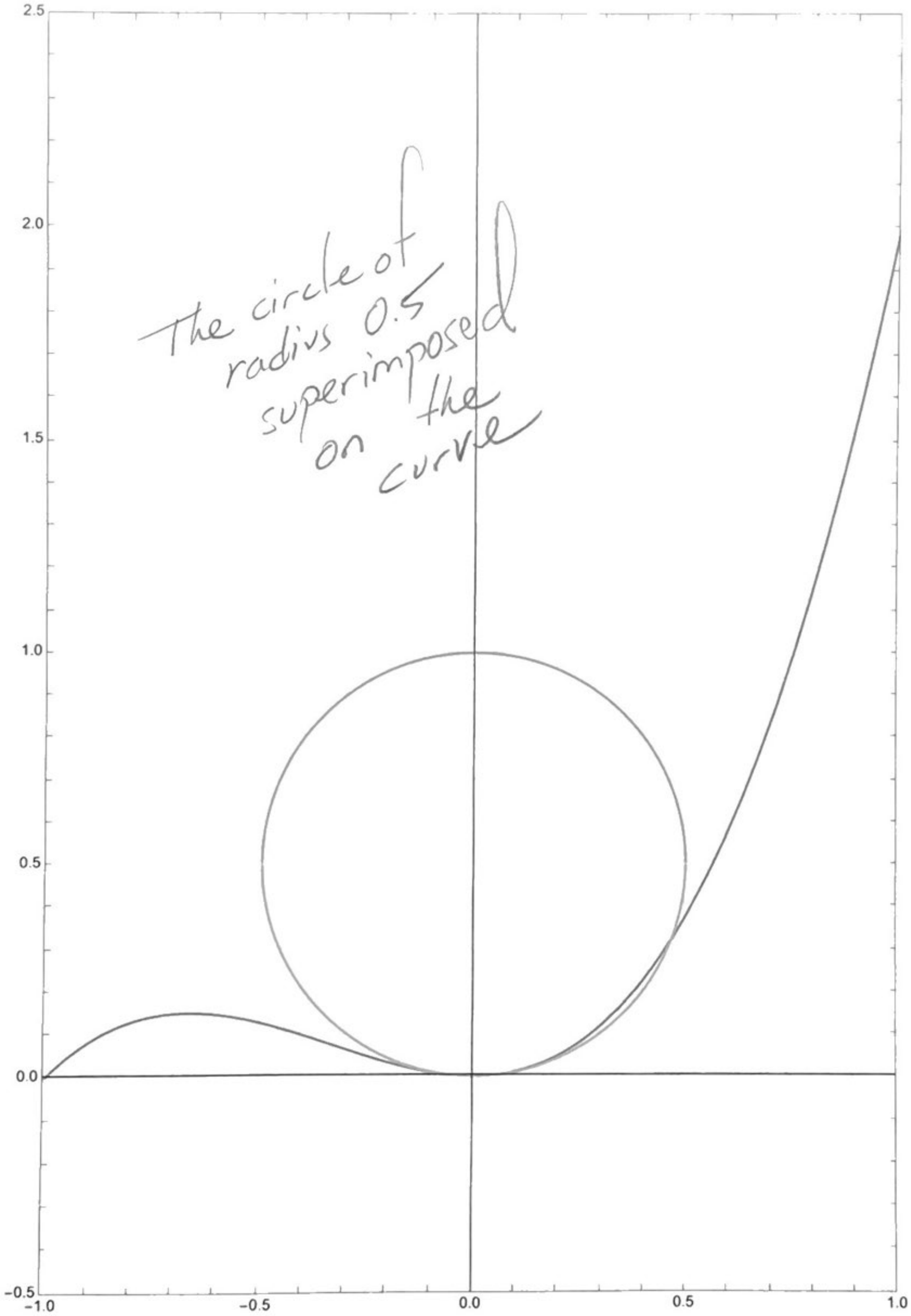
Convergence looks at $x=0$ to 1.0

In[25]:=
```
Plot[x^3 + x^2, {x, -1, 1}, PlotRange → {{-1, 1}, {-0.5, 2.5}}, AspectRatio → 3 / 2,
  GridLines → {Range[-1, 1, 0.1], Range[-0.5, 2.5, 0.1]}, Frame → True]
```

Out[25]=



```
Plot[{x³ + x², -Sqrt[0.25 - x²] + 0.5, Sqrt[0.25 - x²] + 0.5},
 {x, -1, 1}, PlotRange → {{-1, 1}, {-0.5, 2.5}}, AspectRatio → 3 / 2,
 GridLines → {Range[-1, 1, 0.1], Range[-0.5, 2.5, 0.1]}, Frame → True]
```



3. Period Associated with the Van Der Waals Force

$$F_r = -m\frac{v^2}{r} = -m\frac{(2\pi r/T)^2}{r} = -4\pi^2 m\frac{r}{T^2}$$

$$= 12\epsilon\left[\left(\frac{\sigma}{r}\right)^{13} - \left(\frac{\sigma}{r}\right)^{7}\right]/\sigma$$

Multiply through by T^2 and divide through by the $12\epsilon\left[\left(\frac{\sigma}{r}\right)^{13} - \left(\frac{\sigma}{r}\right)^{7}\right]/\sigma$ mess to get

$$T^2 = \frac{-4\pi^2 m\, r\, \sigma}{12\epsilon\left[\left(\frac{\sigma}{r}\right)^{13} - \left(\frac{\sigma}{r}\right)^{7}\right]} \quad \text{or}$$

$$T = \sqrt{\frac{4m \quad r/\sigma}{12\epsilon\left[\left(\frac{\sigma}{r}\right)^{7} - \left(\frac{\sigma}{r}\right)^{13}\right]}}\ \pi\sigma$$

This is only valid for $r > \sigma$. For $r < \sigma$ what is in the square root is negative.

The real Van Der Waal's system is quantum mechanical. I don't know if this period (or frequency) plays any role in its analysis.

It was just intended to be practice with the methods leading up to Corollary 7.