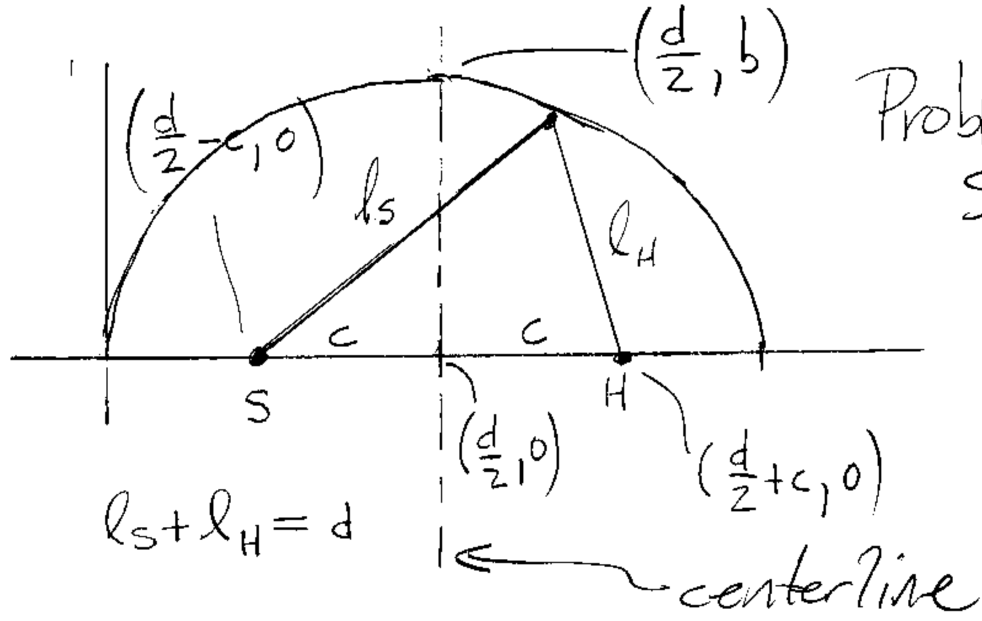
PROPERTIES OF ELLIPSES

Problem Set 5
Solution



1(a) For the point $(0,0)$ the distance to S, $l_S = \frac{d}{2} - c$, and the distance to H, $l_H = \frac{d}{2} + c$.

The sum of these is indeed d, so $(0,0)$ is part of the ellipse

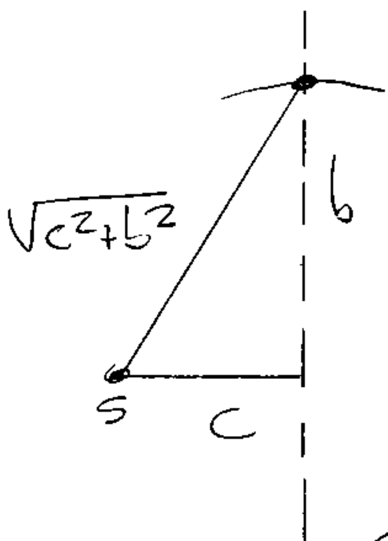
(b) The distance from $(d,0)$ to S at $\left(\frac{d}{2} - c, 0\right)$ is $\frac{d}{2} + c$.

The distance from $(d,0)$ to H at $\left(\frac{d}{2} + c, 0\right)$ is $\frac{d}{2} - c$.

Again, the sum of these is d, so the point $(d,0)$ is part of the ellipse.

1(c) $\left(\frac{d}{2}, b\right)$ is l_S from $\left(\frac{d}{2}-c, 0\right)$

where $l_s = \sqrt{c^2+b^2}$



by the Pythagorean theorem.

The same is true for l_H.

So $2\sqrt{c^2+b^2} = d$

(d) Solve $2\sqrt{c^2+b^2} = d$ for b.

$$b = \sqrt{\left(\frac{d}{2}\right)^2 - c^2} < \frac{d}{2}$$

(e) $$p \equiv \frac{(\text{minor axis})^2}{\text{major axis}} = \frac{(2b)^2}{d}$$

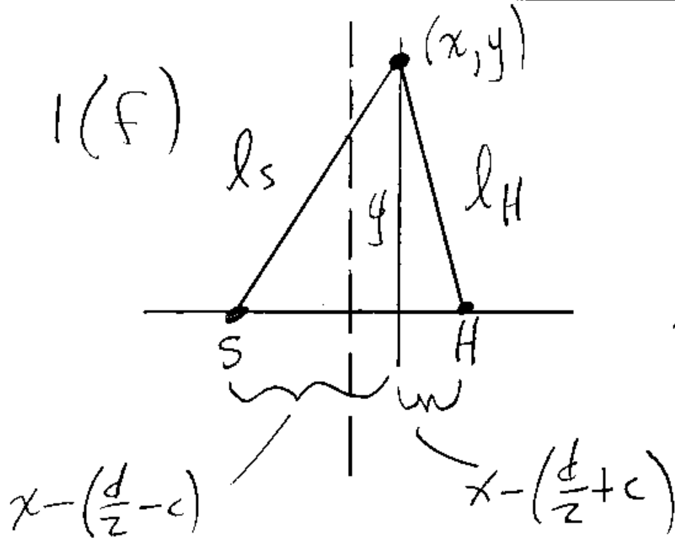
$$\Longrightarrow \quad b = \frac{\sqrt{pd}}{2}$$

Apply this to $2\sqrt{c^2+b^2} = d$

$$\Rightarrow 2\sqrt{c^2 + \frac{pd}{4}} = d \Longrightarrow \sqrt{4c^2+pd} = d$$

$$\Longrightarrow 4c^2 = d^2 - pd$$

1(f)



$$l_s = \sqrt{y^2 + \left(x - \left(\frac{d}{2} - c\right)\right)^2}$$

$$l_H = \sqrt{y^2 + \left(x - \left(\frac{d}{2} + c\right)\right)^2}$$

(g) The problem description says to simplify the LHS of

$$l_H^2 - l_s^2 - d^2 = -2 l_s d$$

as much as we can. So here we go:

$$l_H^2 - l_s^2 - d^2 = \cancel{y^2} + \left(x - \left(\frac{d}{2}+c\right)\right)^2 - \left[\cancel{y^2} + \left(x - \left(\frac{d}{2} - c\right)\right)^2\right] - d^2$$

$$= \cancel{x^2} - 2x\left(\frac{d}{\cancel{2}} + c\right) + \left(\frac{d}{2}+c\right)^2$$
$$- \left[\cancel{x^2} - 2x\left(\frac{d}{\cancel{2}} - c\right) + \left(\frac{d}{2} - c\right)^2\right] - d^2$$

$$= -4xc + \cancel{\frac{d^2}{4}} + cd + \cancel{c^2} - \left(\cancel{\frac{d^2}{4}} - cd + \cancel{c^2}\right)$$
$$- d^2$$

$$= -4xc + 2cd - d^2$$

^ Pretty simple. There is hope ☺

1(h) Square equation again. The LHS is

$$\left(-4xc+2cd-d^2\right)^2$$
$$=16x^2c^2-16xc^2d+8xcd^2$$
$$+4c^2d^2-4cd^3+d^4$$

The RHS is

$$4l_s^2d^2=4\left[y^2+\left(x-\left(\tfrac{d}{2}-c\right)\right)^2\right]d^2$$

(i) Equate LHS=RHS

$$16x^2c^2-16xc^2d\cancel{+8xcd^2}\cancel{+4c^2d^2}\cancel{-4cd^3}\cancel{+d^4}$$
$$=4y^2d^2+4x^2d^2-8x\left(\tfrac{d}{2}\cancel{-c}\right)d^2+\underbrace{4\left(\tfrac{d}{2}-c\right)^2d^2}_{\cancel{d^4}\cancel{-4d^3c}\cancel{+4c^2d^2}}$$

Indeed, the terms linear in c cancelled.

(j) What is left

$$16x^2c^2-16xc^2d=4y^2d^2+4x^2d^2-4xd^3$$

It is suggested that we use $4c^2=d^2-pd$

$$4x^2(\cancel{d^2}-pd)-4x(\cancel{d^2}-pd)d$$
$$=4y^2d^2+\cancel{4x^2d^2}\cancel{-4xd^3}$$
$$-\cancel{4}x^2p\cancel{d}+\cancel{4}xpd^{\cancel{2}}=\cancel{4}y^2d^{\cancel{2}}\Longrightarrow y^2=x\left(p-\frac{xp}{d}\right)$$

2(a) $QV^2 = PV.VR$ $\quad x = PV$

$y^2 = x.VR$ $\quad y = QV$

But VR is unknown, call it ρ. So $y^2 = x\rho$. We need a formula for ρ.

When $PV = x = 0$, clearly $\rho = VR$ becomes PL. When $PV = x = d$, clearly $\rho = VR$ is zero.

$\rho = VR$ linearly decreases from PL to zero as x goes from 0 to d.

The function that does this is

$$\rho = PL\left(1 - \frac{x}{d}\right)$$

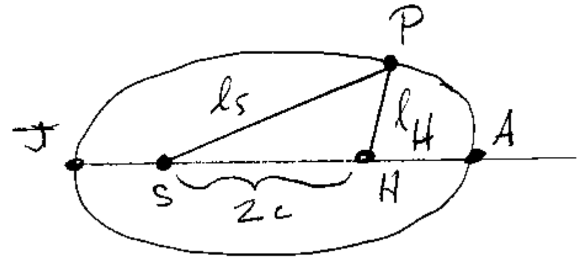
However, PL = latus rectum = p.

So $\rho = p\left(1 - \frac{x}{d}\right)$

(b) Put this into $y^2 = x\rho$, and we get

$$y^2 = xp\left(1 - \frac{x}{d}\right)$$ which is the equation at the bottom of p.12

3\.

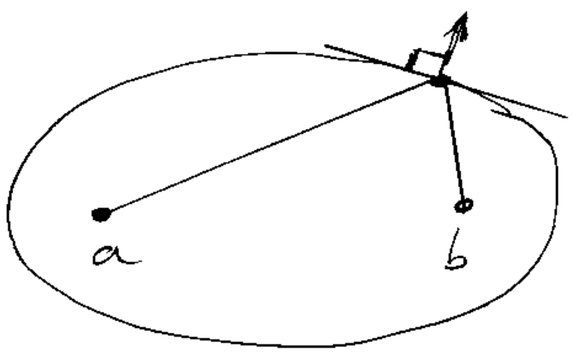


$\overline{PS}+\overline{PH} = d =$ the sum of $l_S + l_H$.

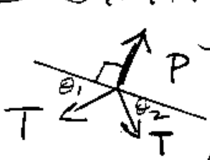
But we already showed that $(0,0)$ and $(0,d)$ are on the ellipse. Therefore $\overline{JS}+\overline{SH}+\overline{HA} = d = 2\overline{AC}$

There $\overline{PS}+\overline{PH} = d = 2\overline{AC}$

4\.



The pencil is being pushed perpendicularly to the tangent. It is also being pulled toward the foci by the tension in the string:



By balancing of forces $\overbrace{P = T\sin\theta_1 + T\sin\theta_2}^{\text{perp to tangent}}$

and $\underbrace{0 = T\cos\theta_1 - T\cos\theta_2}_{\text{parallel to tangent}} \Rightarrow \theta_1 = \theta_2$