Newton — Problem Set 5 — Properties of Ellipses

Due Tuesday, Nov. 1 (beginning of class).

We will take as our definition of the ellipse a very physical one:

Put two points in the plane (these points are the *foci*, and *foci* is the plural of *focus*), and call those points S and H. Choose any distance d provided d is greater than the distance between S and H. Then an ellipse is the set of points whose distance from S, call that l_S, and whose distance from H, call that l_H, adds up to d. In equations $l_S + l_H = d$. Since d is a distance, obviously it must be greater than 0 in what follows.

For definiteness let us place S at coordinates $\left(\frac{d}{2} - c,\ 0\right)$ and H at coordinates $\left(\frac{d}{2} + c,\ 0\right)$. Assume with no loss of generality that $c > 0$.

We can describe what we have so far as two foci that are on the x-axis, placed symmetrically around the point $\left(\frac{d}{2},\ 0\right)$, and separated by an amount $2\,c$.

Note that because $d > 2\,c$, it follows that $\frac{d}{2} - c$ is a positive number. In other words, both foci are on the positive x-axis.

NB: It is really going to help to start making a high-quality drawing of what is being described. This is just an ellipse on the plane. You don't need to attempt anything three-dimensional involving cones.

1. The Basic Formula

Starting with the definition above, you will derive the formula at the bottom of p. 12 of Apollonius.

(a) Show that the point $(0,\ 0)$ is in the set of points comprising the ellipse.

(b) Show that the point $(d,\ 0)$ is in the set of points comprising the ellipse.

NB: By parts (a) and (b), we see that we are using d exactly as it was used in our Apollonius reading.

(c) Draw a vertical line through the point $\left(\frac{d}{2}, 0\right)$. There is a point $\left(\frac{d}{2}, b\right)$ on this centerline that is part of the ellipse. For this point to be on the ellipse, what is the equation involving b, c, and d?

(d) In (a) and (b) we discovered that the major axis of the ellipse has length d. We have just discovered the length of the minor axis. It is $2\,b$. Mishel asked whether it is always true that $2\,b < d$? Show it.

(e) Apollonius finds the latus rectum, p, to be convenient for describing an ellipse. Recall that by definition, $p = \frac{(\text{minor axis})^2}{\text{major axis}} = \frac{(2\,b)^2}{d}$. This means that wherever we see b in part (c), we can replace it with $\frac{1}{2}\sqrt{p\,d}$. Do that and then solve for $4\,c^2$.

(f) Choose a general point $(x,\ y)$ in the plane. Write down formulas for l_S and l_H.

(g) Square the equation $l_S + l_H = d$. You get $l_S{}^2 + l_H{}^2 + 2\,l_S\,l_H = d^2$. Rewrite $2\,l_S\,l_H$ as $2\,l_S(d - l_S)$. So now you have $l_S{}^2 + l_H{}^2 + 2\,l_S(d - l_S) = d^2$ or $l_H{}^2 - l_S{}^2 - d^2 = -2\,l_S\,d$. Plug in the formulae for l_S and l_H that you got in (f) and simplify $l_H{}^2 - l_S{}^2 - d^2$ as much as you can. HINT: $l_H{}^2 - l_S{}^2 - d^2$ simplifies down to just three terms involving only $x\,c$, $c\,d$, and d^2.

(h) Now square the equation again. The RHS is $4\,l_S{}^2\,d^2$. The LHS is the square of whatever you got in (g).

(i) Start simplifying. HINT: The original geometrical situation was invariant under $c \to -c$, so the first things that should disappear are any terms that are linear in c. Once you have shown that those cancel go on to (j).

(j) Keep simplifying. HINT: Now the only way that c appears always involves $4\,c^2$, and you can trade in every occurrence of $4\,c^2$ for the combination of p and d that you found in (e).

FINISHING: Parts (i) and (j) took me an entire page, but we know where we are going and all the smoke has to clear until we get the pleasantly simple formula at the bottom of p. 12 of Apollonius. If you make mistakes, re-copy. Unlike in the days of Newton, paper is plentiful. Parts (i) and (j) are an exercise in doing algebra clearly and accurately as well as a derivation of the pleasantly simple formula for an ellipse.

2. The Basic Formula Again

It is not obvious that $y^2 = p\,x - \frac{p}{d}\,x^2$ is equivalent to $QV^2 = PV.VR$. It is a simple identification that $QV = y$. Additionally, $PV = x$ is also a simple identification. So we have $y^2 = x.VR$. Let’s give VR a name, ρ. So $y^2 = x\,\rho$ is what Apollonius has shown.

(a) By studying the figure on p. 11 of Apollonius, write down a formula for ρ. Your formula for ρ will involve only x, p, and d. This is a good opportunity to make sure that your Greek letter ρ and the Latin letter p are distinct-looking. The Greek letter ρ is made by starting at the bottom of the descender, in a single stroke. The Latin letter p is made with two strokes.

(b) Put what you found for ρ in (a) into $y^2 = x\,\rho$. You should get the pleasantly simple formula at the bottom of p. 12 of Apollonius.

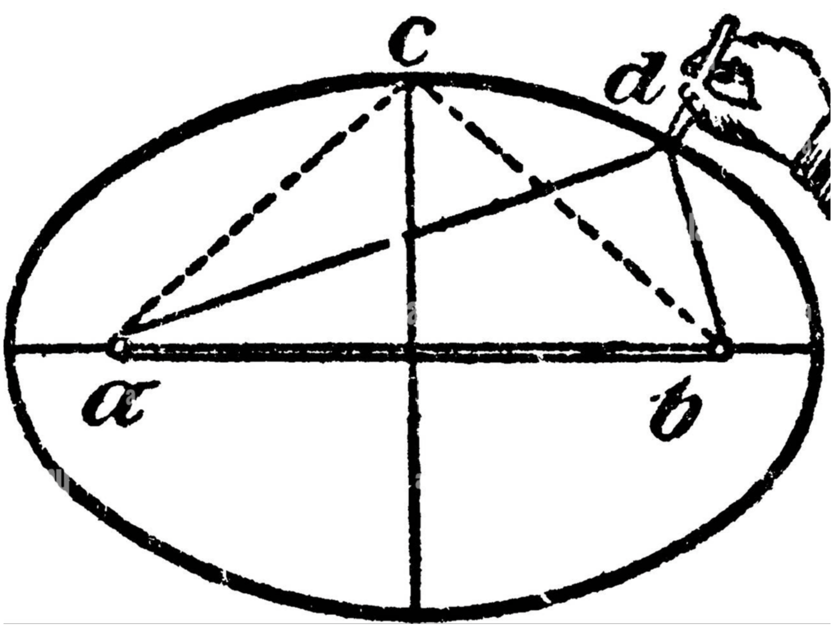
3. An Easy Property of Ellipses used by Newton

In Proposition 11, Newton said "*PS*, *PH* which together are equal to the whole axis 2*AC*." Write a short and simple proof (about two sentences) that demonstrates this. You should include a diagram with the relevant points labeled.

4. A Harder Property used by Newton

In Proposition 11, Newton needed "the equal angles *IPR*, *HPZ*." That is Apollonious III.48. Nutzzzz. It would be nice to be fully convinced that these angles are equal without getting deep into the third book of Apollonius. Here we go....

Consider the definition of the ellipse. Can you see that it precisely equivalent to what is being done in this drawing?



The string connecting *ad* and *bd* is kept taut. Imagine that the string slips easily around the pencil. The tension in the *ad* portion of the string and the *bd* portion of the string are then equal. You keep the string taut and located at *d* by pushing perpendicularly to the tangent at *d*.

Argue, by balancing of forces, that for the pencil to not accelerate in the direction parallel to the tangent that the angles made between the *ad* portion of the string and the tangent, and the *bd* portion of the string and the tangent must be equal. Include a diagram with the forces on the pencil, the angles, and the tangent well-labeled.

This demonstrates "the equal angles *IPR*, *HPZ*."