Newton — Problem Set 8 — Planetary Motion and Lunar Acceleration

Due Thursday, Dec. 1 (beginning of class)

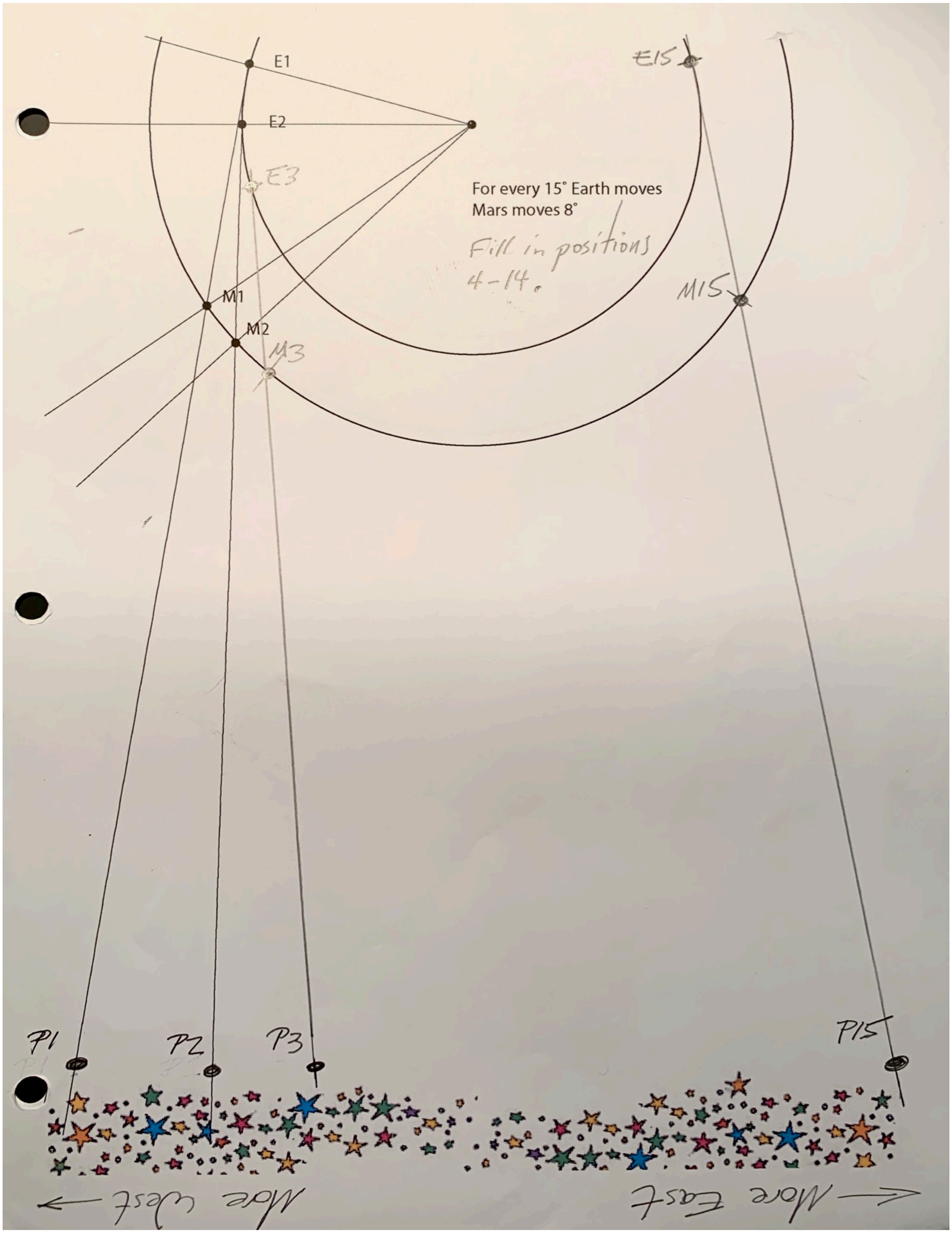
1. The Modern Explanation of Retrograde Motion

In class we discussed the Ptolemaic explanation of retrograde motion (which requires epicycles). The Earth spins around once a day, and of course that creates a great deal of apparent motion. The Earth's annual trip around the Sun causes the Sun to steadily move through the zodiacal constellations in a west-to-east direction. It is a tad less than 1° degree of movement in a day, so that after 365.24 days, the Sun does a full 360°, returning to the same position in the zodiac. The point of this paragraph is that the Sun appears to slowly go west-to-east through the stars. Imagine that the Earth stopped going around the Sun, but Mars continued going around the Sun. The motion of Mars would cause it to slowly go west-to-east through the stars, in the same direction as the Sun. This is called prograde motion.

Of course, the Earth is not stopped, and in fact, it goes around faster than Mars. The surprise is that sometimes Mars appears to go west-to-east through the stars, but some of the time it appears to go east-to-west. The latter is called retrograde motion. You can easily see the effect with circles. The small amount of eccentricity (wherein the planets are actually traveling on ellipses rather than circles) is completely unnecessary to introduce if the goal is just to see that there is retrograde motion in the Sun-centered (Copernican) model.

Directions:

A. Mars: On the following page, carefully mark equally-spaced movements for Mars. Mars goes about 8° around the Sun every two weeks. You need to measure the spacings and then make additional equal spacings. M1 and M2 were done by computer. M3 was done by hand. You mark positions M4 to M14.
B. Earth: Carefully mark equally-spaced — but larger — movements for Earth. Earth goes further around the Sun than Mars in each two-week period. It's about 15°. By computer, E1 and E2 have been marked. E3 shows how to continue by hand. You mark Earth positions E4 to E14. For steps A and B, the most important thing to get right is equal spacings for each Mars movement and the equal but larger spacings for each Earth movement.
C. The next part of the geometrical construction is to see where Mars appears to be in front of the distant stars. By computer I have shown P1, which is where M1 appears to be in the stars when Earth is at E1, and P2, which where M2 appears to be when Earth is at E2. By hand, but using a straight edge, I have shown P3, which is where M3 appears to be when Earth is at E3.
D. You continue by doing P4 to P14 with a straight-edge, which is where Mars positions M4 to M14 appear to be when Earth is at the corresponding positions E4 to E14.


E1
E2
E3
For every 15° Earth moves
Mars moves 8°
Fill in positions
4-14.
E15
M1
M2
M3
M15
P1
P2
P3
P15
More West
More East

1. The Modern Explanation of Retrograde Motion (Continued)

E. Now that you have completed the construction, hold it up so that the stars are along the ecliptic. In other words, face south, and tilt the paper up so that looking from your vantage point here on Earth, the stars appear high in the sky (like Jupiter is in the evenings right now). Holding the construction up this way, you should easily be able to complete this statement: Most of the time Mars moves ___________________________________ through the stars (and this is called prograde motion).
F. Still holding the construction up to get your bearings, complete this statement: For a brief time, Mars moves ________________________________ through the stars (and this is called retrograde motion).
G. Each step in the construction represents two weeks, so: Mars is retrograde for about ______________ weeks.

2. Determining the Distance to Jupiter

Nowadays we can get the distance to objects in the solar system by bouncing radar off of them.

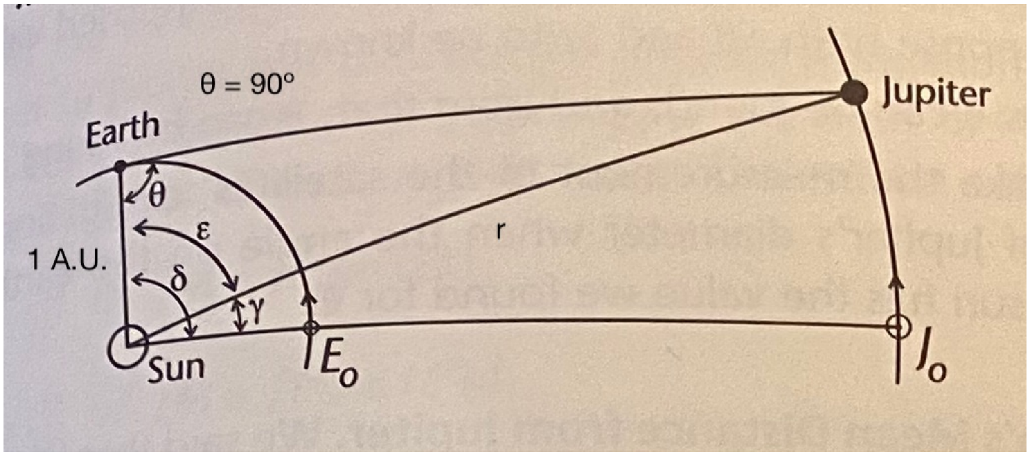
Of course, Newton didn't have radar (nor did any other pre-20th-century astronomers). It turns out you can't get the distance to the planets without some major trickery involving a "transit of Venus." Very good measurements using that were done in 1769, but prior to that, all people could get were the relative sizes of the planetary orbits.

We assume that the size of Earth's orbit (average radius) is 1. A.U. (where "A.U." is short for "Astronomical Unit").

Can we get the size of Jupiter's orbit in terms of the A.U.? The answer is of course yes, otherwise there would be no way to build the tables in Phenomenon 4. Note that Earth has the arbitrary value "100.000" on p. 336. That's Newton's way of capturing the ignorance of the A.U.

On pp. 310-311 Densmore described the method. You will need the figure on p. 311, which I have reproduced on the next page.

2. Determining the Distance to Jupiter (Continued)



A. On September 26 of this year, Earth, Jupiter, and the Sun were aligned in "opposition." (Which means that the Sun and Jupiter were in opposite directions in the sky — which also means that you couldn't see them at the same time, because as the Sun set, Jupiter rose.

B. Now the job is to wait until Jupiter is 90° degrees from the Sun. This point is known as "eastern quadrature." This is going to occur on Dec. 22nd this year.

C. The number of days elapsed is 30 (days hath September) + 31 (days hath October) + 30 (days hath November) + 22 - 26 = 87 days.

D. It is known from longer duration observations that the complete orbital period of Jupiter is 4333 days. If you can't see how you would determine this, that is a good problem to contemplate.

E. So Earth has gone 87/365 of the way around the Sun, and Jupiter has gone 87/4333 of the way around the Sun.

F. Convert those fractions to degrees. Earth has gone $\delta =$ ________ degrees, while Jupiter has gone $\gamma =$ ________ degrees. We only have two significant figures in our counting of the days, so there is definitely no need to keep more than one decimal place.

G. So you know $\epsilon = \delta - \gamma =$ ________ degrees.

H. Examining the figure, what trig function and equation involves r, 1 A.U., and ϵ?

I. Solve for r.

J. Plug in, using the value for ϵ that you got in G. What is the value of r? (Make sure you calculator is in degrees mode.)

K. Since Newton prefers to capture the ignorance of the A.U. as "100.000," using his units, what is the value of r? Newton's table on p. 336 goes to six significant figures, but there are disagreements among the experimenters already at the 1 part in 1,000 level. Our crude calculation where we rounded to the nearest day should agree to about 1 part in 50. To do better, we'd have to know what hour on September 26th Jupiter was in opposition, and what hour on December 22nd it will be in eastern quadrature, and also take into account the extra hour "fall back" if the times are quoted in typical local time.

3. Comparing q_E and q_M

The Moon is 60x as far away from the center of the Earth as the surface of the Earth is. If gravity ("terrestrial heaviness") is also what is keeping the Moon in orbit, and if the force law that pulls things toward the center of the Earth goes as $1/r^2$, then the acceleration of the Moon should be 1/3600 of that of a rock dropped here on Earth.

We wish to recreate Newton's Proposition 4 argument for ourselves, using algebra (which is entirely equivalent to what Newton does, but is more compact, and more importantly, more understandable and memorable to a modern reader).

(a) A formula for q_E

We define: $q_E \equiv$ distance fallen starting from rest / (time elapsed)^2 here on the surface of the Earth.

Newton did not have sufficiently good timers to measure q_E, and even if he did, experimenters of his time could get a better result by using a pendulum. Newton cites (and later proves!) the relationship between q_E and the period of a pendulum. It is:

$$T = 2\pi\sqrt{\frac{L}{2\,q_E}}$$

L is the length of the pendulum. Solve this equation for q_E.

(b) A formula for q_M

We define: $q_M \overset{\text{ult}}{\equiv}$ distance deflected / (time elapsed)^2 by the Moon in its orbit in the limit where the distance deflected and the elapsed time are both very small. The distance deflected I call d. The perpendicular distance I call $d_\perp$. The time elapsed and the perpendicular distance are related by:

time elapsed $= d_\perp / v$

Furthermore, $v = C/P$, where C is the circumference of the Moon's orbit, and P is the period of the Moon's orbit, which is 27.3 days (if you want, ask me why it isn't the value commonly cited of 29.5 days). Converting to minutes, $P = 39\,343$ minutes (I didn't actually convert — I grabbed this off of Densmore p. 366). Now is not the time to plug in P. The point is just that P is known. Finally, C is known (it is 60 times the circumference of the Earth, because the Moon orbits 60 times as far away as the surface, and the French have measured the circumference of the Earth).

Use all of these facts to get a formula for q_M that involves only d, $d_\perp$, C, and P.

(c) Getting rid of d and $d_\perp$

The equation of a circle of radius R is $y^2 = 2x\left(R - \frac{x}{2}\right)$. You should check for yourself (as I did in class), that $(0, 0)$, $(2R, 0)$, (R, R), and $(R, -R)$, are all satisfied by this equation. For (x, y) very near $(0, 0)$, you can neglect $\frac{x}{2}$ compared to R where those two terms appear in the parenthesis. So you are left with $y^2 = 2xR$.

When (x, y) are very near $(0, 0)$, we have been identifying y as $d_\perp$ and x as d.

Make those identifications in $y^2 = 2xR$ and then use the result to get rid of d and $d_\perp$ in the formula for q_M that you found in part (b).

(d) Plugging in numbers

It is a convenient (but fundamentally unimportant) coincidence that the radius of the Moon's orbit is 60 Earth radii and that 60 is the number of seconds in a minute.

Newton exploits this coincidence (and we will exploit it too) as follows: If you compute q_M in the units of feet/minute^2 and you compute q_E in the units of feet / second^2, *and if you find that they agree* in these disparate units, then you have shown that the acceleration of gravity at the Moon is 1/3600 of the acceleration of gravity here on Earth. This is the stunning confirmation of the $1/r^2$ force law Newton is claiming in Proposition 4.

Compute q_M and q_E using the numbers that Newton has on hand. You can use Paris feet or English feet, as long as you are consistent.

If you don't like Newton's units trick, convert everything to (Paris or English) feet / second^2 and just show that q_E agrees with $3600\, q_M$.

You should mentally contemplate whether the amount of agreement is better, about the same, or worse than expected.