Newton — Problem Set 9 — The Ring Theorem and Tides

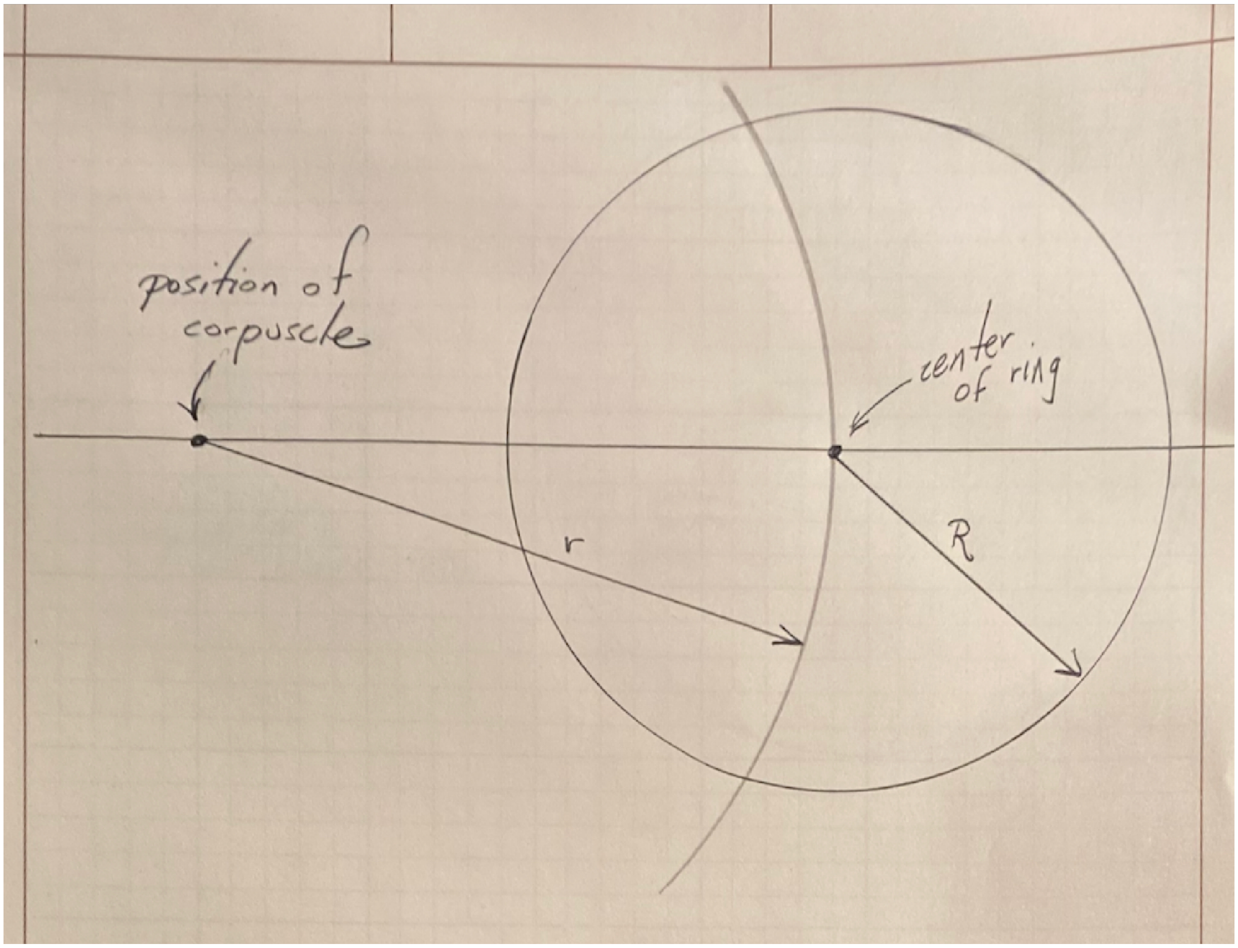
Due Friday, Dec. 9 (beginning of class)

Problem 1 — "The Ring Theorem" — Mathematical Experimentation

Newton's Shell Theorem requires you to think in 3-d and attempt to make quality representations of 3-d shapes on a 2-d page, which isn't easy. There is an analog of the shell theorem in 2-d which I will call "The Ring Theorem," and it is almost as interesting while being far easier to represent on a 2-d page.

In 2-d, gravity falls as $\frac{1}{r}$ instead of $\frac{1}{r^2}$. If a corpuscle has mass m, and a rotationally-symmetric disk has mass M, the 2-d ring theorem claims that the force between them is simply $\frac{GmM}{r}$, where r is the distance from the corpuscle to *the center* of the disk.

Just as in the Shell Theorem, we decompose the rotationally-symmetric disk into rotationally-symmetric, infinitesimally-thin rings, and make an argument about each ring. Let's do some experimental mathematics with one of these rings. We are going to turn it into beads! In the limit that the number of beads goes to infinity, it is a rotationally-symmetric ring. Perhaps a ring composed of 36 beads each spaced 10° around the ring will behave fairly closely to a rotationally-symmetric ring.

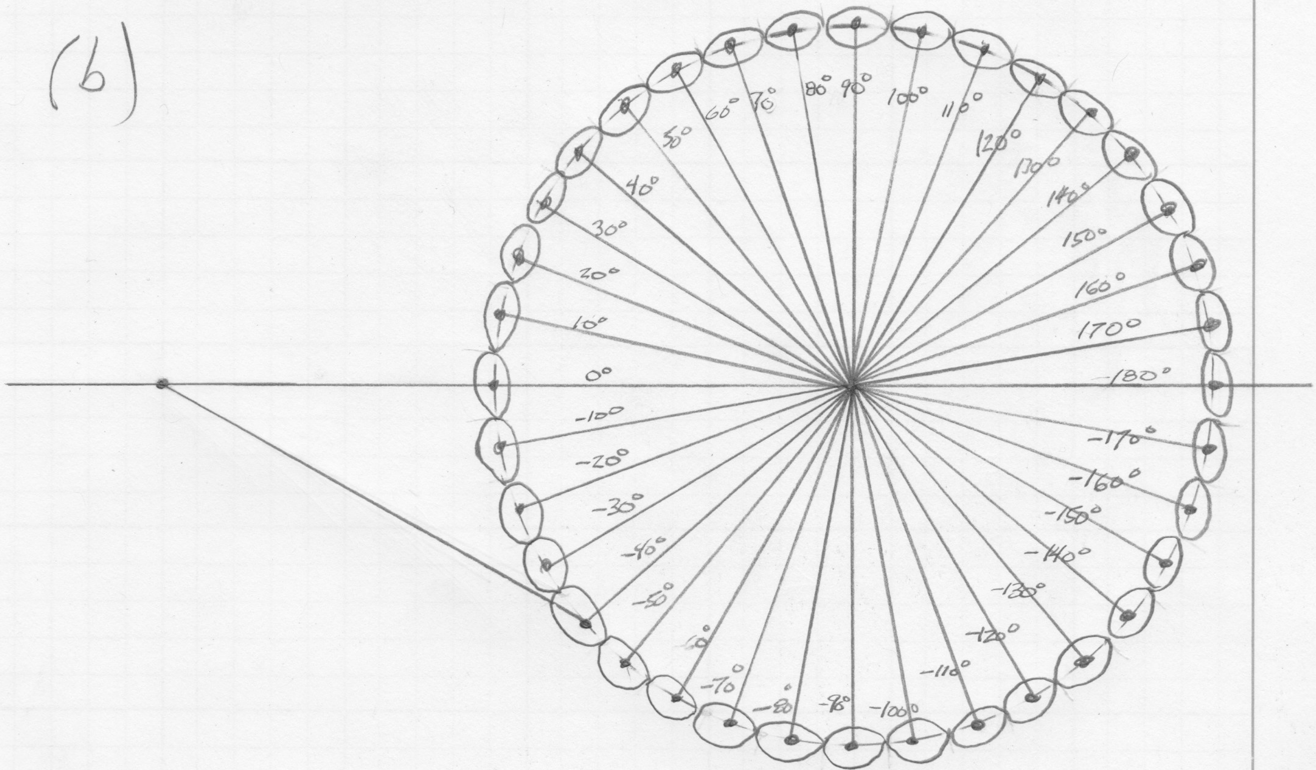


(a) A faint arc of radius *r* has been added to the figure above. Estimate as a percentage, what fraction of the ring is closer than *r* to the corpuscle? And what percentage is farther than *r*?

Note: Isn't it interesting and a bit magical that all the bits of the ring that are at a distance less than *r* (and compose less than half of the ring) are pulling just enough harder than "their share" to just balance all the bits that are farther than *r* (which compose more than half of the ring) and which are each pulling less than their share?!!

On the next page I have drawn a ring that has been broken up into 36 beads and made some measurements from the drawing.

(b)



Distance from corpuscle to center of ring = 91.5

we'll do everything in mm

Distance from corpuscle to center of bead at $-40° = r_{-40°} = 64.5$

Angle of force of bead at $-40°$ relative to horizontal $= \theta_{-40°} = 29°$

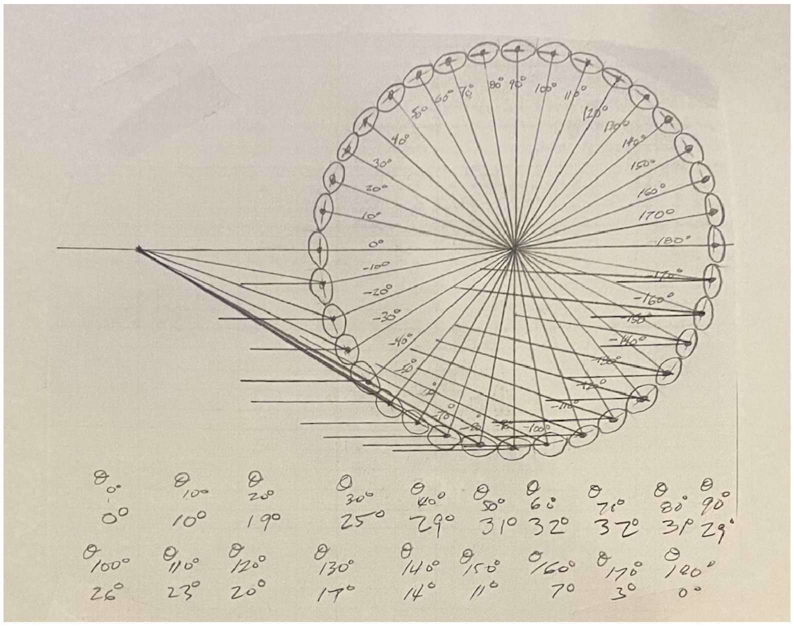
Cosine of this angle $= \cos\theta_{-40°} = 0.875$

Product of $\cos\theta_{-40°} \cdot \frac{1}{r_{-40°}} = 0.0136$

The boxed work has to be done for all 36 beads. However, the bead at $-40°$ has the same value as the one at $+40°$ (etcetera), so there are only 19 measurements to do!

1(b) (Continued)

Note that in the table on the following page, I have already done all the angle measurements for you. Below is the drawing I used to get them. Be sure you understand what angles I measured and why we are taking their cosine.



Also note that the bottom half of the table you are about to build has the same values as the top half (in reverse order), so once you have done all the beads in half of the ring, including the bead at 0° and the bead at 180°, you don't have to measure and calculate any more. You can just copy the corresponding values.

Using the diagram on the previous page, and being as accurate as I have been (measurements to nearest half millimeter), and keeping four decimal places in the last column, fill in the table on the following page.

After you do that, return to 1(c).

1 (c) Now for the moment of truth. Total up the final column and divide by 36. What do you get?

How close is it to 1/91.5 = 0.0109 which is what the ring theorem leads us to expect?

If there is very close agreement, congratulations, you have "experimentally" demonstrated the ring theorem. If your agreement is not good, figure out what went wrong.

Out[89]//TableForm=

$\theta_{-170°}$	$-3°$	$\cos\theta_{-170°}$	0.99863	r		$\frac{\cos\theta_{-170°}}{r}$	
$\theta_{-160°}$	$-7°$	$\cos\theta_{-160°}$	0.992546	r		$\frac{\cos\theta_{-160°}}{r}$	
$\theta_{-150°}$	$-11°$	$\cos\theta_{-150°}$	0.981627	r		$\frac{\cos\theta_{-150°}}{r}$	
$\theta_{-140°}$	$-14°$	$\cos\theta_{-140°}$	0.970296	r		$\frac{\cos\theta_{-140°}}{r}$	
$\theta_{-130°}$	$-17°$	$\cos\theta_{-130°}$	0.956305	r		$\frac{\cos\theta_{-130°}}{r}$	
$\theta_{-120°}$	$-20°$	$\cos\theta_{-120°}$	0.939693	r		$\frac{\cos\theta_{-120°}}{r}$	
$\theta_{-110°}$	$-23°$	$\cos\theta_{-110°}$	0.920505	r		$\frac{\cos\theta_{-110°}}{r}$	
$\theta_{-100°}$	$-26°$	$\cos\theta_{-100°}$	0.898794	r		$\frac{\cos\theta_{-100°}}{r}$	
$\theta_{-90°}$	$-29°$	$\cos\theta_{-90°}$	0.87462	r		$\frac{\cos\theta_{-90°}}{r}$	
$\theta_{-80°}$	$-31°$	$\cos\theta_{-80°}$	0.857167	r		$\frac{\cos\theta_{-80°}}{r}$	
$\theta_{-70°}$	$-32°$	$\cos\theta_{-70°}$	0.848048	r		$\frac{\cos\theta_{-70°}}{r}$	
$\theta_{-60°}$	$-32°$	$\cos\theta_{-60°}$	0.848048	r		$\frac{\cos\theta_{-60°}}{r}$	
$\theta_{-50°}$	$-31°$	$\cos\theta_{-50°}$	0.857167	r		$\frac{\cos\theta_{-50°}}{r}$	
$\theta_{-40°}$	$-29°$	$\cos\theta_{-40°}$	0.87462	r	64.5	$\frac{\cos\theta_{-40°}}{r}$	0.0136
$\theta_{-30°}$	$-25°$	$\cos\theta_{-30°}$	0.906308	r		$\frac{\cos\theta_{-30°}}{r}$	
$\theta_{-20°}$	$-19°$	$\cos\theta_{-20°}$	0.945519	r		$\frac{\cos\theta_{-20°}}{r}$	
$\theta_{-10°}$	$-10°$	$\cos\theta_{-10°}$	0.984808	r		$\frac{\cos\theta_{-10°}}{r}$	
θ_{0}	0	$\cos\theta_{0}$	1.	r		$\frac{\cos\theta_{0}}{r}$	
$\theta_{10°}$	$10°$	$\cos\theta_{10°}$	0.984808	r		$\frac{\cos\theta_{10°}}{r}$	
$\theta_{20°}$	$19°$	$\cos\theta_{20°}$	0.945519	r		$\frac{\cos\theta_{20°}}{r}$	
$\theta_{30°}$	$25°$	$\cos\theta_{30°}$	0.906308	r		$\frac{\cos\theta_{30°}}{r}$	
$\theta_{40°}$	$29°$	$\cos\theta_{40°}$	0.87462	r	64.5	$\frac{\cos\theta_{40°}}{r}$	0.0136
$\theta_{50°}$	$31°$	$\cos\theta_{50°}$	0.857167	r		$\frac{\cos\theta_{50°}}{r}$	
$\theta_{60°}$	$32°$	$\cos\theta_{60°}$	0.848048	r		$\frac{\cos\theta_{60°}}{r}$	
$\theta_{70°}$	$32°$	$\cos\theta_{70°}$	0.848048	r		$\frac{\cos\theta_{70°}}{r}$	
$\theta_{80°}$	$31°$	$\cos\theta_{80°}$	0.857167	r		$\frac{\cos\theta_{80°}}{r}$	
$\theta_{90°}$	$29°$	$\cos\theta_{90°}$	0.87462	r		$\frac{\cos\theta_{90°}}{r}$	
$\theta_{100°}$	$26°$	$\cos\theta_{100°}$	0.898794	r		$\frac{\cos\theta_{100°}}{r}$	
$\theta_{110°}$	$23°$	$\cos\theta_{110°}$	0.920505	r		$\frac{\cos\theta_{110°}}{r}$	
$\theta_{120°}$	$20°$	$\cos\theta_{120°}$	0.939693	r		$\frac{\cos\theta_{120°}}{r}$	
$\theta_{130°}$	$17°$	$\cos\theta_{130°}$	0.956305	r		$\frac{\cos\theta_{130°}}{r}$	
$\theta_{140°}$	$14°$	$\cos\theta_{140°}$	0.970296	r		$\frac{\cos\theta_{140°}}{r}$	
$\theta_{150°}$	$11°$	$\cos\theta_{150°}$	0.981627	r		$\frac{\cos\theta_{150°}}{r}$	
$\theta_{160°}$	$7°$	$\cos\theta_{160°}$	0.992546	r		$\frac{\cos\theta_{160°}}{r}$	
$\theta_{170°}$	$3°$	$\cos\theta_{170°}$	0.99863	r		$\frac{\cos\theta_{170°}}{r}$	
$\theta_{180°}$	0	$\cos\theta_{180°}$	1.	r		$\frac{\cos\theta_{180°}}{r}$	

Problem 2 — “The Ring Theorem” Proved

Follow Newton’s Shell Theorem carefully, making it your own, and — instead of documenting that — instead write down the corresponding steps, equations, and drawings for the slightly simpler ring theorem, again making it your own.

Problem 3 — The Tides

(a) The mass of the Moon is M =7.348 x 10^{22} kilograms. The constant G in the gravitational force formula $\frac{GMm}{r^2}$ is 6.674 x10^{-11} $m^3/kg/s^2$. So for the Moon, GM= 4.904 x10^{12} m^3/s^2. The average distance from the Moon to the Earth is r = 3.844 x10^8 m. Square r and calculate $\frac{GM}{r^2}$ to get the average acceleration of the Earth due to the Moon. The units are m/s^2.

(b) The mass of the Sun is M =1.989 × 10^{30} kilograms. So for the Sun, GM= 1.327 x10^{20} m^3/s^2. The average distance from the Sun to the Earth is r = 1.474 x10^{11} m. Square r and calculate $\frac{GM}{r^2}$ to get the average acceleration of the Earth due to the Sun. As a cross-check, your answer to (b) will be roughly 200x your answer to (a).

(c) The near part of the Earth is about 59/60 as far from the Moon as the center of the Earth, and the far part of the Earth is about 61/60 as far from the Moon as the center. Using your answer to (a) and these distance ratios, find the acceleration of the part of the Earth nearest to the Moon and the acceleration of the part of the Earth farthest from the Moon. The Earth’s crust is a thick pile of rock, and perhaps one wouldn’t expect it to be very deformable. However, the water on the surface of the Earth is certainly deformable. How much more is the water on the near side of the Earth being accelerated toward the Moon than the water on the far side of the Earth?

(d) The near part of the Earth is about 23140/23141 as far from the Sun as the center of the Earth and the far part of the Earth is about 23142/23141 as far from the Sun at the center. Repeat the methodology in part (c), this time using the acceleration calculated in part (b). Because $(23\,141/23\,140)^2$ and $(23\,141/23\,142)^2$ are both so close to 1, you are going to have to be creative to get a good answer to the question “how much more is the water on the near side of the Earth being accelerated toward the Sun than the water on the far side of the Earth?”

1. Do you see why the tides that are raised by the Moon bulge on both the near and far side of the Earth?
2. By the way, the acceleration formulae we have used are entirely correct, but they aren’t enough by themselves to tell you how high the tides are raised. They just tell you that they are raised — on both sides of the Earth — and they also tell you how much larger the effect of the Moon is than the Sun in raising tides.
3. Do you see why the tides raised by the Moon can be about twice the size of the tides raised by the Sun, despite the fact that the Sun’s gravity is far more powerful?

4. Does it seem there should be four high tides per day: two caused by the Moon and two caused by the Sun? I assure you there are only two high tides each day. In fact, the sum of the tidal forces from two bodies is just a tidal force whose peaks are on a new line that goes through neither the Moon nor the Sun (but because the Moon's effect is stronger, the line is closer to the Moon).

5. As one final point, when the Moon and the Sun are at 90° to one another (as in first quarter or third quarter Moon), they weaken the total effect, and when they are aligned *or opposite* to each other (as in new Moon or full Moon), their effects add. All this is simultaneously deep and simple stuff.

Below is a tide chart for Santa Barbara through the end of the semester: