Term 3 Exam — Solution

1. Tides — The Roche Limit

(a) $\frac{Gm}{r^2}$

(b) $\frac{GM}{R^2}$

(c) $\frac{GM}{(R-r)^2}$

per the suggested approximation, the first term is $\approx 1 + \frac{2r}{R}$

(d) $\frac{GM}{R^2}\left[\frac{1}{(1-r/R)^2} - 1\right] \approx \frac{2GMr}{R^3}$

(e) $\frac{\cancel{G}m}{r^2} = \frac{2\cancel{G}Mr}{R^3} \Rightarrow \frac{2M}{m} = \frac{R^3}{r^3}$

(f) If $M = 500{,}000m$, the LHS is 1,000,000.
The cube root of 1,000,000 is 100, so

$$\frac{R}{r} = 100 \quad \text{or} \quad R = 100r$$

If the moon gets closer than 100r to the planet with 500,000x the mass, rocks are lifted off its surface.

2. The Methods of the Pendulum Applied to the Spring

(a) $\dfrac{m_1}{m_2} = \dfrac{k_1}{k_2} \dfrac{T_1^2}{T_2^2}$

(b) The quantities of matter in bodies connected to springs are in the ratio compounded of the strengths of the springs and the duplicate ratio of the times of oscillation. (in a vacuum)

(c) For the velocity which a given force can generate in a given matter in a given time is as the force and the time directly, and the matter inversely $\left(\dfrac{v_1}{v_2} = \dfrac{k_1}{k_2} \dfrac{t_1}{t_2} \dfrac{m_2}{m_1}\right)$.

But now, the motive forces in places equidistant from equilibrium are as the strengths of the springs, and therefore if two bodies in oscillating describe equal displacements, and those displacements be divided into equal parts, since the times in the parts are as the times of the whole, the velocities in the parts will be as the motive forces and the whole times, directly and the matter inversely. And consequently, the quantities of matter are as the forces and the times of oscillation directly and the velocities inversely. But the velocities are inversely as the times, and consequently the times directly and the velocities inversely are as the squares of the times, and therefore the quantities of matter are as the motive forces and the squares of the times, that is as the strengths of the springs and the squares of the times.

Q.E.D.

3. The Shell Theorem in 4-d

(a) $$\frac{F_2}{F_1} = \frac{m_2}{m_1}\,\frac{1/i_2^3}{1/i_1^3}\,\frac{\cos\theta_2}{\cos\theta_1}$$

(b) $$\frac{F_2}{F_1} = \frac{a_2}{a_1}\,\frac{s_2^2}{s_1^2}\,\frac{1/i_2^3}{1/i_1^3}\,\frac{\cos\theta_2}{\cos\theta_1}$$

(c) $$\frac{F_2}{F_1} = \frac{c_2}{c_1}\,\frac{s_2^2}{s_1^2}\,\frac{1/i_2^3}{1/i_1^3}\,\frac{\cos\theta_2}{\cos\theta_1}$$

(d) $$\frac{F_2}{F_1} = \frac{f_1}{f_2}\,\frac{s_2^2}{s_1^2}\,\frac{1/i_2^{\cancel{3}\,2}}{1/i_1^{\cancel{3}\,2}}\,\frac{\cos\theta_2}{\cos\theta_1}$$

(e) $$\frac{F_2}{F_1} = \frac{f_1}{f_2}\left(\frac{r_1}{r_2}\right)^2\frac{\cos\theta_2}{\cos\theta_1}$$

(f) But $f_1/\cos\theta_1 = r_1$ and $f_2/\cos\theta_2 = r_2$, so

$$\frac{F_2}{F_1} = \frac{r_1^3}{r_2^3}$$ woo HOO!

(g) Re-derive $\frac{c_2}{c_1} = \frac{i_2}{i_1}\frac{f_1}{f_2}$ by contemplating d, f_1, i_1, and c_1

$$\frac{d}{f_1} = \frac{c_1}{i_1} \Rightarrow d = \frac{f_1 c_1}{i_1}$$

also $d = \frac{f_2 c_2}{i_2}$ equate the two expressions for d

$$\frac{f_1 c_1}{i_1} = \frac{f_2 c_2}{i_2} \Rightarrow \frac{c_2}{c_1} = \frac{i_2}{i_1}\,\frac{f_1}{c_2}$$

(h) Re-derive $\frac{s_2}{s_1}\frac{i_1}{i_2} = \frac{r_1}{r_2}$ by contemplating e, r_1, s_1, and i_1

$$\frac{e}{r_1} = \frac{s_1}{i_1} \Rightarrow e = \frac{r_1 s_1}{i_1}$$

also $e = \frac{r_2 s_2}{i_2}$ equate the two expressions for e

$$\frac{r_1 s_1}{i_1} = \frac{r_2 s_2}{i_2} \Rightarrow \frac{s_2}{s_1}\,\frac{i_1}{i_2} = \frac{r_1}{r_2}$$