Newton — Term 3 Exam

Monday, Dec. 12

I expect that the last problem is a fair amount harder than the first two. Budget something like 15, 15, and 30 minutes for the three problems, respectively (rather than 20, 20, and 20).

1. Tides — The Roche Limit (7 pts)

Imagine a small moon (small in both radius and mass), orbiting a large planet. I have drawn the moon with a rock sitting on its surface, on the side facing the planet.

(a) Write an expression for the acceleration of the rock towards the moon, due to the gravitational pull of the moon. This formula should only involve, G, m, and r.

(b) Write an expression for the acceleration of the moon towards the planet, due to the gravitational pull of the planet. This formula should only involve G, M, and R.

(c) The rock on the near side of the moon is closer to the planet by the amount r. Write an expression for the acceleration of the rock towards the planet, due to the gravitational pull of the planet, that takes into account the fact that it is only $R - r$ from the center of the planet.

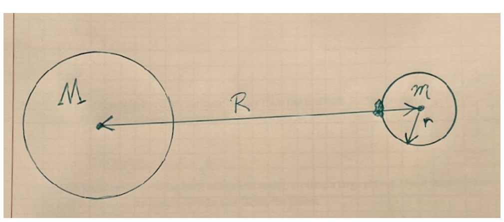
(d) We want to know how much more the planet is accelerating the rock vs. the moon as a whole. So, subtract the acceleration in (b) from the acceleration in (c), but also make the following approximation, and there will be a lot of simplification:

$$\frac{1}{(R-r)^2} = \frac{1}{R^2}\,\frac{1}{(1-r/R)^2} \approx \frac{1}{R^2}\left(1+\frac{2r}{R}\right)$$

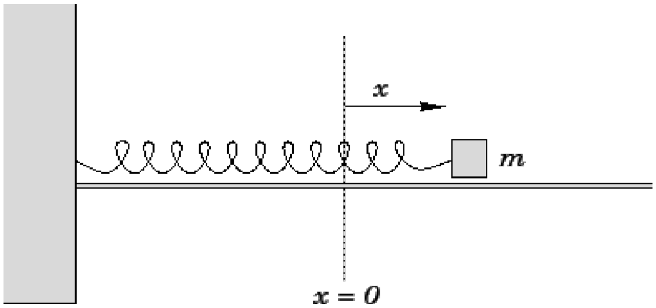
(e) Set the acceleration you got in (d) equal to the acceleration you got in (a). This is telling you when the tidal force of the planet on the moon is just strong enough to lift rocks off the surface of the moon.

(f) If $M = 500\,000\,m$, what does the relationship in (e) simplify to as a relationship between R and r?

Note: If the moon gets closer than the radius R that you found in (f), unless it has a lot of internal strength and very few loose rocks, it will be torn apart. This is called the Roche limit.



2. The Methods of the Pendulum Applied to the Spring (6 pts)



The mass on a spring is entirely analogous to but simpler than a weight on a pendulum. Here is what Newton said in Book II, Proposition 24 about a weight on a pendulum:

Book II Proposition 24

The quantities of matter in pendulous bodies whose centers of oscillation are equally distant from the center of suspension are in the ratio compounded of the ratio of the weights and the duplicate ratio of the times of oscillation in a vacuum.

For the velocity which a given force can generate in a given matter in a given time is as the force and the time directly, and the matter inversely. To the extent that the force is greater or the time is greater or the matter is less, a greater velocity will be generated. This is manifest through the Second Law of Motion. But now if the pendulums are of the same length, the motive forces in places equidistant from the perpendicular are as the weights, and therefore if two bodies in oscillating describe equal arcs, and those arcs be divided into equal parts, since the times in which the bodies describe the individual corresponding parts of the arcs are as the times of the whole oscillations, the velocities in the corresponding parts of the oscillations will be to each other as the motive forces and the whole times of the oscillations directly and the quantities of matter inversely, and consequently the quantities of matter are as the forces and the times of oscillations directly and the velocities inversely. But the velocities are inversely as the times, and consequently the times directly and the velocities inversely are as the squares of the times, and therefore the quantities of matter are as the motive forces and the squares of the times, that is, as the weights and the squares of the times.

Q.E.D.

(a) The mathematical statement Newton is making is $\frac{m_1}{m_2} = \frac{w_1}{w_2} \frac{T_1^2}{T_2^2}$.

Make the corresponding mathematical statement of the Proposition but for the spring.

HINT: Instead of the restoring force being caused by a weight dangling at various angles, the restoring force is caused by the stretching or compressing of the spring from its equilibrium position. So wherever Densmore discussed weight, w, and the restoring force, $w \sin\theta$, you instead have the strength of the spring, k, and the restoring force kx.

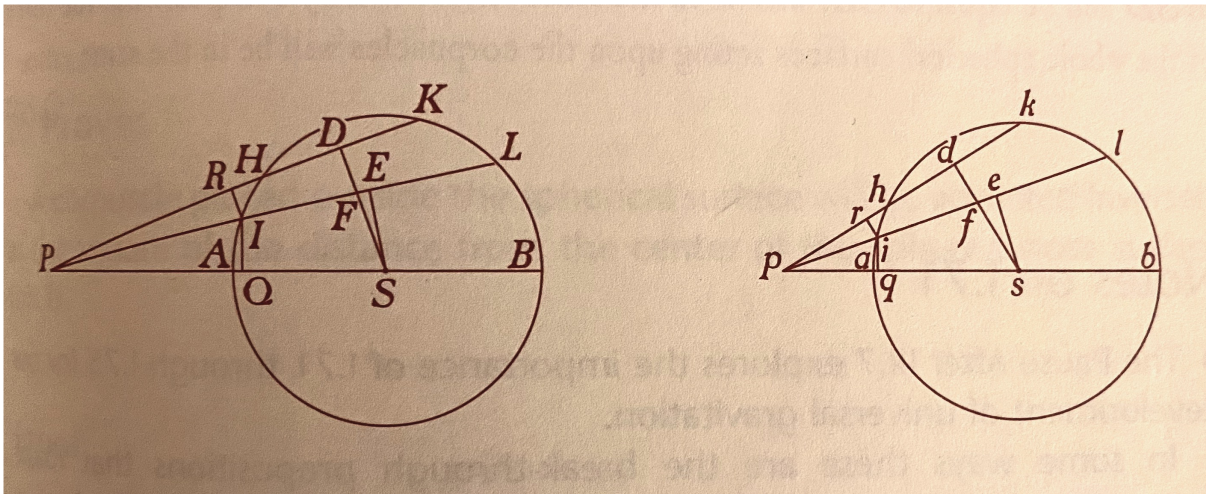
(b) Translate the mathematical statement in (a) into words, as Newton does in his opening sentence. Your statement should begin: “The quantities of matter in bodies connected to springs are in the ratio compounded of ....”

HINT: There is nothing in the spring problem that corresponds to the length from the center of suspension. So you can just ignore Newton’s comments that involve that length. That is why the spring problem is analogous to but simpler than the pendulum problem.

(c) Now that you have made the claim for the spring corresponding to Newton’s claim for the pendulum, write a proof corresponding to Newton’s. Be terse.

3. The Shell Theorem in Four Spatial Dimensions (12 pts)

Note: If you prefer Newton's notation, feel free to use it. My diagrams are on the next page. Here are Donahue's:



Newton did the shell theorem in three spatial dimensions, and on the last problem set, you did it for the simpler case of two spatial dimensions. With those cases under your belt, the obvious case to try next is four spatial dimensions. In four spatial dimensions the universal law of gravitation is:

$$F = \frac{GMm}{r^3}$$

The cross-sectional drawings are the exact same as in two and three dimensions, but the full situation represented by the drawings is no longer possible to visualize. In two dimensions, the mass residing in arc a_2 divided by the mass residing in arc a_1, was simply:

$$\frac{m_2}{m_1} = \frac{a_2}{a_1}$$

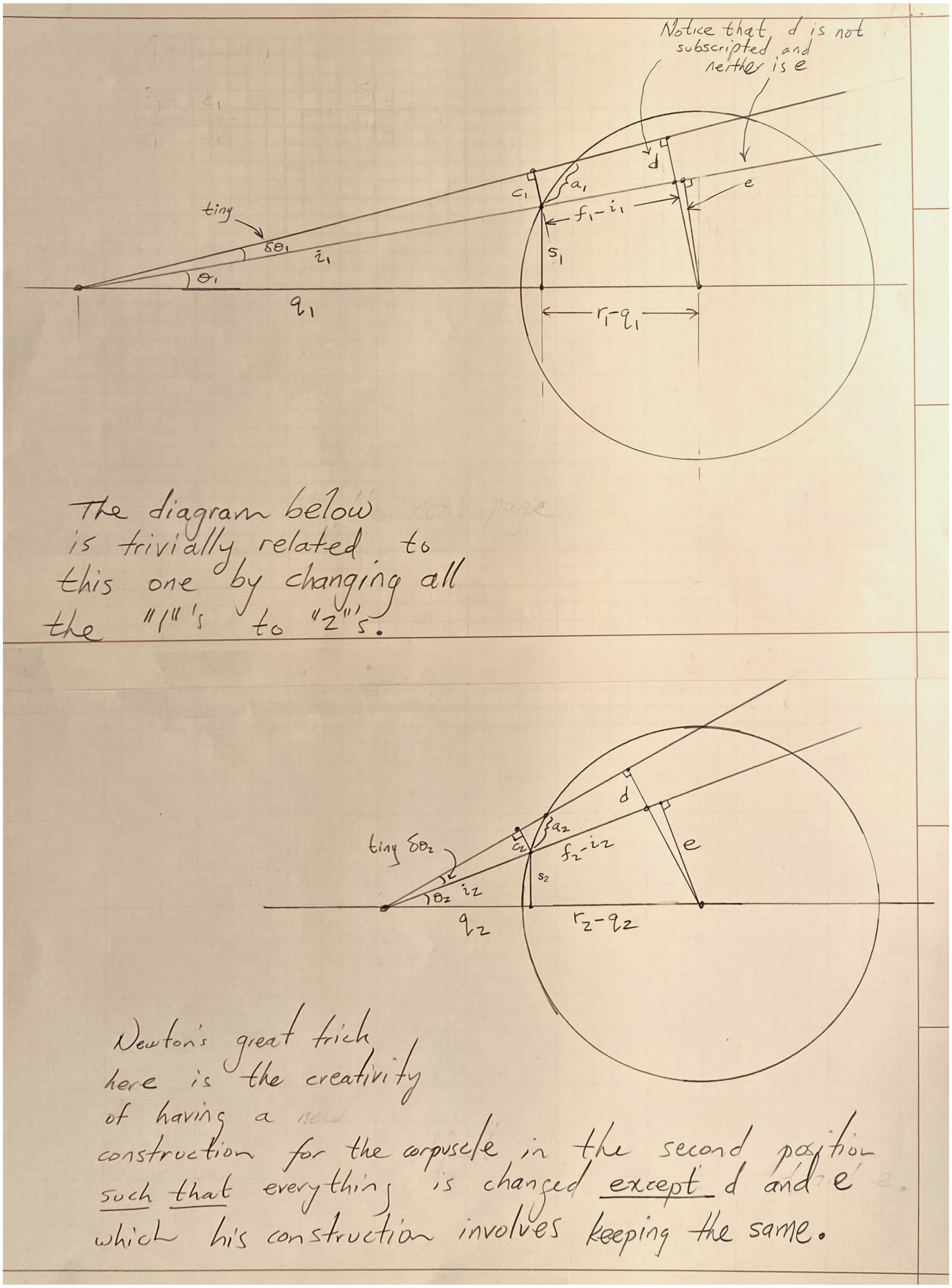
In three dimensions, the mass residing in the hoop by revolving a_2 divided by the mass residing in the hoop by revolving a_1, was:

$$\frac{m_2}{m_1} = \frac{a_2}{a_1} \frac{2\pi s_2}{2\pi s_1} = \frac{a_2}{a_1} \frac{s_2}{s_1}$$

In four dimensions, although you can't readily visualize it, the ratio of m_2/m_1 is:

$$\frac{m_2}{m_1} = \frac{a_2}{a_1} \frac{4\pi s_2^2}{4\pi s_1^2} = \frac{a_2}{a_1} \frac{s_2^2}{s_1^2}$$

Oh, so you just get one more power of $\frac{s_2}{s_1}$ in the formula for $\frac{m_2}{m_1}$ each time you go up a dimension. Simple!



NB: In my diagrams above all of the lower-case Latin letters are lengths. Upper-case *F* is distinct from lower-case *f*. In the discussion on the next page, F_1 and F_2 are forces.

(a) Write down the starting point in four dimensions. In three dimensions it was:

$$\frac{F_2}{F_1} = \frac{m_2}{m_1} \frac{1/i_2^2}{1/i_1^2} \frac{\cos\theta_2}{\cos\theta_1}$$

(Just write it down. There is no need for a derivation because all the ratios are straightforward.)

(b) The next thing we did in either two or three dimensions was apply a formula to get rid of $\frac{m_2}{m_1}$. Apply the four-dimensional formula for $\frac{m_2}{m_1}$ to what you wrote down in (a):

$$\frac{m_2}{m_1} = \frac{a_2}{a_1} \frac{s_2^2}{s_1^2}$$

(c) Now get rid of $\frac{a_2}{a_1}$ using:

$$\frac{a_2}{a_1} = \frac{c_2}{c_1}$$

(d) Then get rid of $\frac{c_2}{c_1}$ using:

$$\frac{c_2}{c_1} = \frac{i_2}{i_1} \frac{f_1}{f_2}$$

(e) There is another formula involving $\frac{i_2}{i_1}$. It was

$$\frac{s_2}{s_1} \frac{i_1}{i_2} = \frac{r_1}{r_2}$$

Since you have two powers of $\frac{s_2}{s_1} \frac{i_1}{i_2}$, you can use this formula to get rid of both of them.

(f) Home stretch: do the remaining geometry and get your final, simple and compelling answer for $\frac{F_2}{F_1}$.

(g) Re-derive the formula that was recited and used in (d). You need to relate d to f_1, i_1, and c_1, and then relate d to f_2, i_2, and c_2, and then equate your two expressions for d.

(h) Re-derive the formula that was recited and used in (e). You need to relate e to r_1, s_1, and i_1, and then relate e to r_2, s_2, and i_2, and then equate your two expressions for e.

Congratulations, you now have proofs for the shell theorem in two, three, and four dimensions, and it is clear how the theorem generalizes to n dimensions.