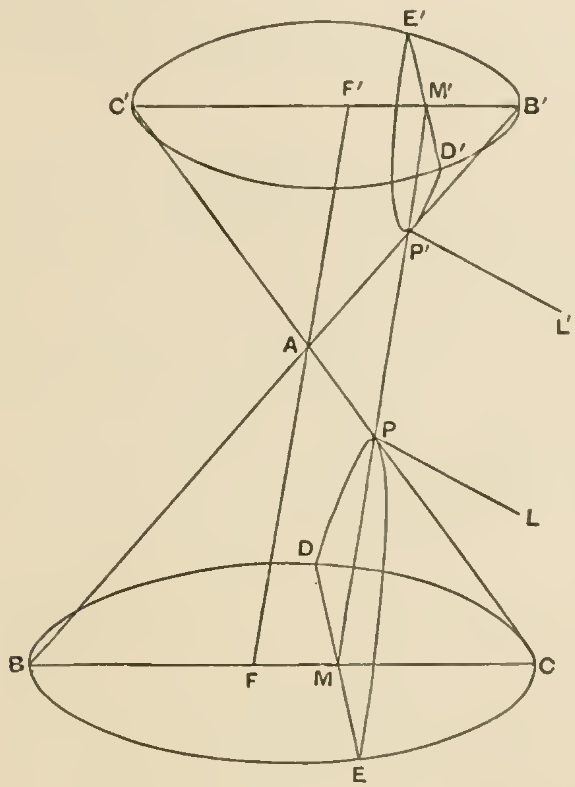
Proposition 4.

[I. 14.]

If a plane cuts both parts of a double cone and does not pass through the apex, the sections of the two parts of the cone will both be hyperbolas which will have the same diameter and equal latera recta corresponding thereto. And such sections are called OPPOSITE BRANCHES.



Let BC be the circle about which the straight line generating the cone revolves, and let $B'C'$ be any parallel section cutting the opposite half of the cone. Let a plane cut both halves of the cone, intersecting the base BC in the straight line DE and the plane $B'C'$ in $D'E'$. Then $D'E'$ must be parallel to DE.

Let BC be that diameter of the base which bisects DE at right angles, and let a plane pass through BC and the apex A cutting the circle $B'C'$ in $B'C'$, which will therefore be a diameter of that circle and will cut $D'E'$ at right angles, since $B'C'$ is parallel to BC, and $D'E'$ to DE.

Let FAF' be drawn through A parallel to MM', the straight line joining the middle points of DE, $D'E'$ and meeting CA, $B'A$ respectively in P, P'.

Draw perpendiculars PL, $P'L'$ to MM' in the plane of the section and of such length that

$$PL : PP' = BF \,.\, FC : AF^2,$$

$$P'L' : P'P = B'F' \,.\, F'C' : AF'^2.$$

Since now MP, the diameter of the section DPE, when produced, meets BA produced beyond the apex, the section DPE is a hyperbola.

Also, since $D'E'$ is bisected at right angles by the base of the axial triangle $AB'C'$, and $M'P$ in the plane of the axial triangle meets $C'A$ produced beyond the apex A, the section $D'P'E'$ is also a hyperbola.

And the two hyperbolas have the same diameter $MPP'M'$.

It remains to prove that $PL = P'L'$.

We have, by similar triangles,

$$BF : AF = B'F' : AF',$$

$$FC : AF = F'C' : AF'.$$

$$\therefore BF \,.\, FC : AF^2 = B'F' \,.\, F'C' : AF'^2.$$

Hence
$$PL : PP' = P'L' : P'P.$$

$$\therefore PL = P'L'.$$

THE DIAMETER AND ITS CONJUGATE.

Proposition 5.

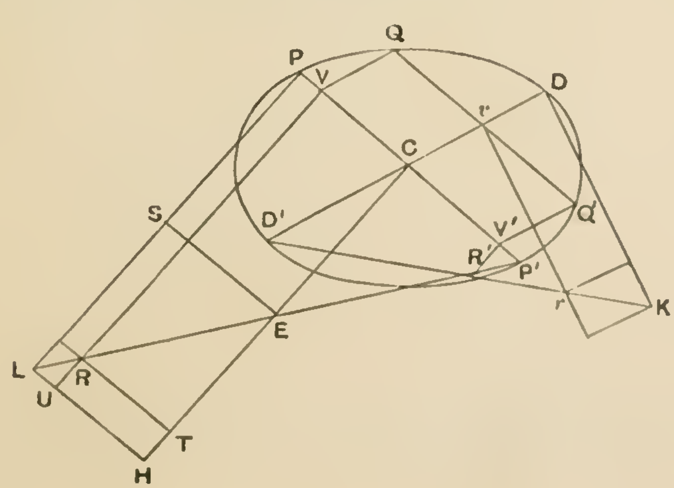
[I. 15.]

If through C, the middle point of the diameter PP′ of an ellipse, a double ordinate DCD′ be drawn to PP′, DCD′ will bisect all chords parallel to PP′, and will therefore be a diameter the ordinates to which are parallel to PP′.

In other words, *if the diameter bisect all chords parallel to a second diameter, the second diameter will bisect all chords parallel to the first.*

Also the parameter of the ordinates to DCD′ will be a third proportional to DD′, PP′.

(1) Let QV be any ordinate to PP', and through Q draw QQ' parallel to PP' meeting DD' in v and the ellipse in Q'; and let $Q'V'$ be the ordinate drawn from Q' to PP'.



Then, if PL is the parameter of the ordinates, and if $P'L$ is joined and VR, CE, $V'R'$ drawn parallel to PL to meet $P'L$, we have [Prop. 3]

$$QV^2 = PV \,.\, VR,$$
$$Q'V'^2 = PV' \,.\, V'R';$$

and $QV = Q'V'$, because QV is parallel to $Q'V'$ and QQ' to PP'.

$$\therefore PV \,.\, VR = PV' \,.\, V'R'.$$

Hence $PV : PV' = V'R' : VR = P'V' : P'V$.

$$\therefore PV : PV' \sim PV = P'V' : P'V \sim P'V',$$

or

$$PV : VV' = P'V' : VV'.$$
$$\therefore PV = P'V'.$$

Also $CP = CP'$.

By subtraction, $CV = CV'$,

and $\therefore Qv = vQ'$, so that QQ' is bisected by DD'.

(2) Draw DK at right angles to DD' and of such a length that $DD' : PP' = PP' : DK$. Join $D'K$ and draw vr parallel to DK to meet $D'K$ in r.

Also draw TR, LUH and ES parallel to PP'.

Then, since $PC = CP'$, $PS = SL$ and $CE = EH$;

$\therefore$ the parallelogram $(PE) = (SH)$.

Also $(PR) = (VS) + (SR) = (SU) + (RH)$.

By subtraction, $(PE) - (PR) = (RE)$;

$$\therefore CD^2 - QV^2 = RT \,.\, TE.$$

But $CD^2 - QV^2 = CD^2 - Cv^2 = D'v \,.\, vD$.

$$\therefore D'v \,.\, vD = RT \,.\, TE \quad \text{..................(A)}.$$

Now $DD' : PP' = PP' : DK$, by hypothesis.

$$\therefore DD' : DK = DD'^2 : PP'^2$$
$$= CD^2 : CP^2$$
$$= PC \,.\, CE : CP^2$$
$$= RT \,.\, TE : RT^2,$$

and

$$DD' : DK = D'v : vr$$
$$= D'v \,.\, vD : vD \,.\, vr;$$
$$\therefore D'v \,.\, vD : Dv \,.\, vr = RT \,.\, TE : RT^2.$$

But $D'v \,.\, vD = RT \,.\, TE$, from (A) above;

$$\therefore Dv \,.\, vr = RT^2 = CV^2 = Qv^2.$$

Thus DK is the parameter of the ordinates to DD', such as Qv.

Therefore the parameter of the ordinates to DD' is a third proportional to DD', PP'.

Cor. We have $$CD^2 = PC \,.\, CE$$
$$= \tfrac{1}{2}PP' \,.\, \tfrac{1}{2}PL;$$
$$\therefore DD'^2 = PP' \,.\, PL,$$
or $$PP' : DD' = DD' : PL,$$
and PL is a third proportional to PP', DD'.

Thus the relations of PP', DD' and the corresponding parameters are reciprocal.

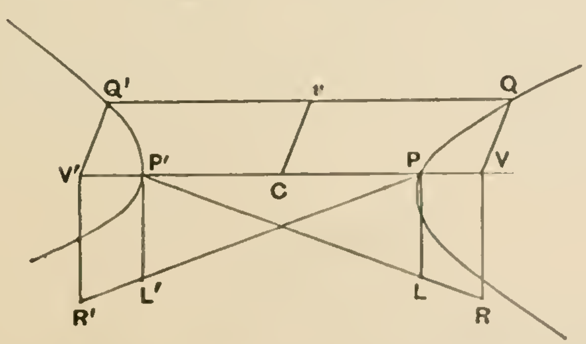
Def. Diameters such as PP', DD', each of which bisects all chords parallel to the other, are called **conjugate diameters**.

Proposition 6.

[I. 16.]

If from the middle point of the diameter of a hyperbola with two branches a line be drawn parallel to the ordinates to that diameter, the line so drawn will be a diameter conjugate to the former one.

If any straight line be drawn parallel to PP', the given diameter, and meeting the two branches of the hyperbola in Q, Q' respectively, and if from C, the middle point of PP', a straight line be drawn parallel to the ordinates to PP' meeting QQ' in v, we have to prove that QQ' is bisected in v.



Let QV, $Q'V'$ be ordinates to PP', and let PL, $P'L'$ be the parameters of the ordinates in each branch so that [Prop. 4]

$PL = P'L'$. Draw VR, $V'R'$ parallel to PL, $P'L'$, and let PL', $P'L$ be joined and produced to meet $V'R'$, VR respectively in R', R.

Then we have $$QV^2 = PV \,.\, VR,$$
$$Q'V'^2 = P'V' \,.\, V'R'.$$

$\therefore\ PV \,.\, VR = P'V' \,.\, V'R'$, and $V'R' : VR = PV : P'V'$.

Also $PV' : V'R' = PP' : P'L' = P'P : PL = P'V : VR$.

$$\therefore\ PV' : P'V = V'R' : VR$$
$$= PV : P'V', \text{ from above;}$$
$$\therefore\ PV' : PV = P'V : P'V',$$

and $$PV' + PV : PV = P'V + P'V' : P'V',$$

or $$VV' : PV = VV' : P'V';$$
$$\therefore\ PV = P'V'.$$

But $$CP = CP';$$

$\therefore$ by addition, $CV = CV'$,

or $$Qv = Q'v.$$

Hence Cv is a diameter conjugate to PP'.

[More shortly, we have, from the proof of Prop. 2,
$$QV^2 : PV \,.\, P'V = PL : PP',$$
$$Q'V'^2 : P'V' \,.\, PV' = P'L' : PP',$$

and $$QV = Q'V',\ PL = P'L';$$

$\therefore\ PV \,.\, P'V = PV' \,.\, P'V'$, or $PV : PV' = P'V' : P'V$,

whence, as above, $PV = P'V'$.]

Def. The middle point of the diameter of an ellipse or hyperbola is called the **centre**; and the straight line drawn parallel to the ordinates of the diameter, of a length equal to the mean proportional between the diameter and the parameter, and bisected at the centre, is called the **secondary diameter** (*δευτέρα διάμετρος*).

Proposition 7.

[I. 20.]

In a parabola the square on an ordinate to the diameter varies as the abscissa.

This is at once evident from Prop. 1.

Proposition 8.

[I. 21.]

In a hyperbola, an ellipse, or a circle, if QV be any ordinate to the diameter PP′,

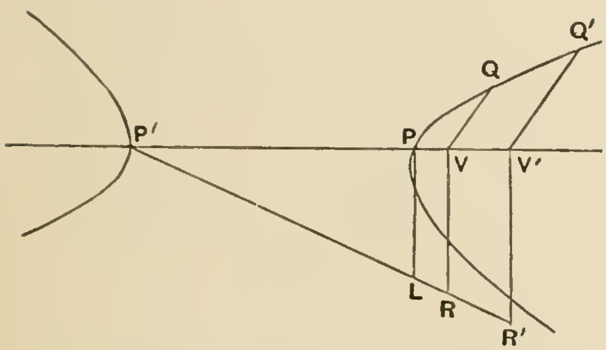
$$QV^2 \propto PV . P'V.$$

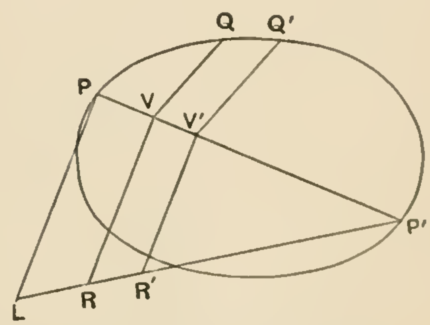
[This property is at once evident from the proportion

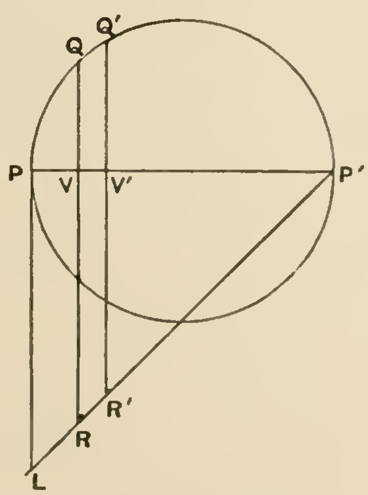
$$QV^2 : PV . P'V = PL : PP'$$

obtained in the course of Props. 2 and 3; but Apollonius gives a separate proof, starting from the property $QV^2 = PV . VR$ which forms the basis of the definition of the conic, as follows.]

Let QV, $Q'V'$ be two ordinates to the diameter PP'.







Then $$QV^2 = PV . VR,$$

$$Q'V'^2 = PV' . V'R';$$

$$\therefore QV^2 : PV . P'V = PV . VR : PV . P'V$$

$$= VR : P'V = PL : PP'.$$

Similarly $Q'V'^2 : PV'.P'V' = PL : PP'$.

$$\therefore\ QV^2 : Q'V'^2 = PV.P'V : PV'.P'V';$$

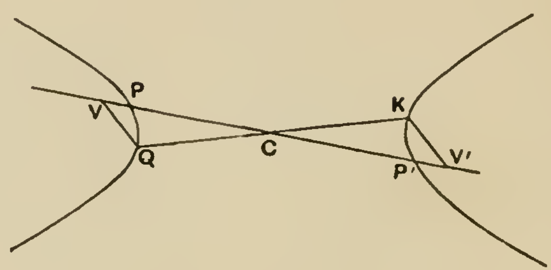
and $QV^2 : PV.P'V$ is a constant ratio,

or $$QV^2 \propto PV.P'V.$$

Proposition 9.

[I. 29.]

If a straight line through the centre of a hyperbola with two branches meet one branch, it will, if produced, meet the other also.



Let PP' be the given diameter and C the centre. Let CQ meet one branch in Q. Draw the ordinate QV to PP', and set off CV' along PP' on the other side of the centre equal to CV. Let $V'K$ be the ordinate to PP' through V'. We shall prove that QCK is a straight line.

Since $CV = CV'$, and $CP = CP'$, it follows that $PV = P'V'$;

$$\therefore\ PV.P'V = P'V'.PV'.$$

But $QV^2 : KV'^2 = PV.P'V : P'V'.PV'$. [Prop. 8]

$\therefore\ QV = KV'$; and QV, KV' are parallel, while $CV = CV'$.

Therefore QCK is a straight line.

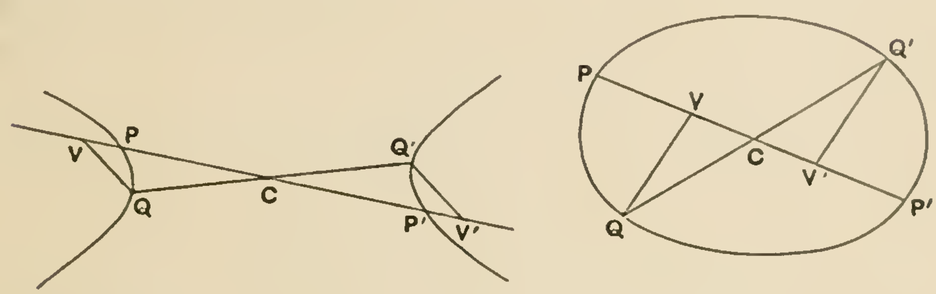
Hence QC, if produced, will cut the opposite branch.

Proposition 10.

[I. 30.]

In a hyperbola or an ellipse any chord through the centre is bisected at the centre.

Let PP' be the diameter and C the centre; and let QQ' be any chord through the centre. Draw the ordinates QV, $Q'V'$ to the diameter PP'.



Then

$$PV.P'V : P'V'.PV' = QV^2 : Q'V'^2$$
$$= CV^2 : CV'^2, \text{ by similar triangles.}$$
$$\therefore CV^2 \pm PV.P'V : CV^2 = CV'^2 \pm P'V'.PV' : CV'^2$$

(where the upper sign applies to the ellipse and the lower to the hyperbola).

$$\therefore CP^2 : CV^2 = CP'^2 : CV'^2.$$

But $$CP^2 = CP'^2;$$

$$\therefore CV^2 = CV'^2, \quad \text{and} \quad CV = CV'.$$

And QV, $Q'V'$ are parallel;

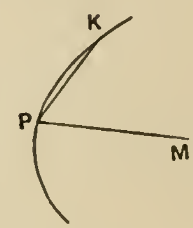
$$\therefore CQ = CQ'.$$

Proposition 11.

[I. 17, 32.]

If a straight line be drawn through the extremity of the diameter of any conic parallel to the ordinates to that diameter, the straight line will touch the conic, and no other straight line can fall between it and the conic.

It is first proved that the straight line drawn in the manner described will fall without the conic.



For, if not, let it fall within it, as PK, where PM is the given diameter. Then KP, being drawn from a point K on the conic parallel to the ordinates to PM, will meet PM and will be bisected by it. But KP produced falls without the conic; therefore it will not be bisected at P.

Therefore the straight line PK must fall without the conic and will therefore touch it.

It remains to be proved that no straight line can fall between the straight line drawn as described and the conic.

(1) Let the conic be a *parabola*, and let PF be parallel to the ordinates to the diameter PV. If possible, let PK fall between PF and the parabola, and draw KV parallel to the ordinates, meeting the curve in Q.

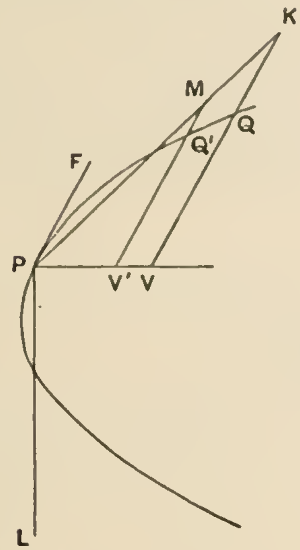
Then
$$\begin{aligned} KV^2 : PV^2 &> QV^2 : PV^2 \\ &> PL . PV : PV^2 \\ &> PL : PV. \end{aligned}$$

Let V' be taken on PV such that

$$KV^2 : PV^2 = PL : PV',$$

and let $V'Q'M$ be drawn parallel to QV, meeting the curve in Q' and PK in M.

Then $KV^2 : PV^2 = PL : PV'$
$$= PL \,.\, PV' : PV'^2$$
$$= Q'V'^2 : PV'^2,$$



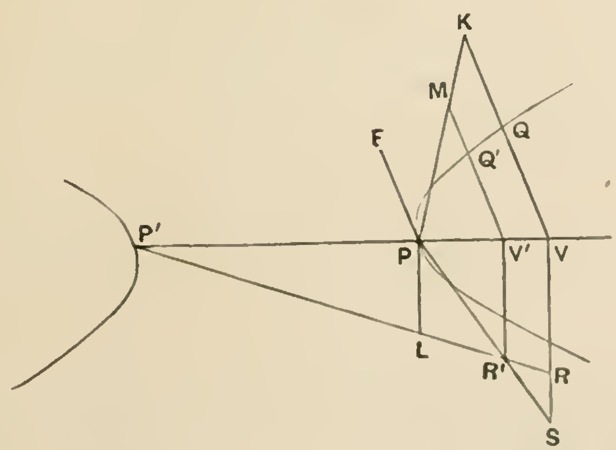
and $\quad KV^2 : PV^2 = MV'^2 : PV'^2$, by parallels.

Therefore $\quad MV'^2 = Q'V'^2$, and $MV' = Q'V'$.

Thus PK cuts the curve in Q', and therefore does not fall outside it: which is contrary to the hypothesis.

Therefore no straight line can fall between PF and the curve.

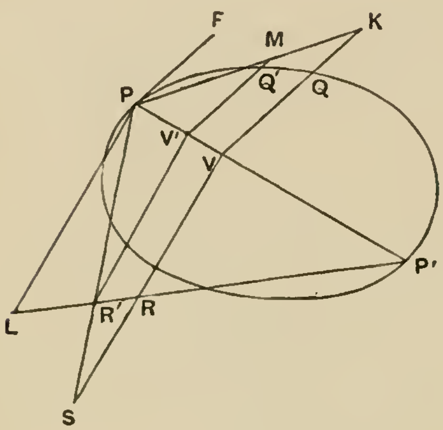
(2) Let the curve be a *hyperbola* or an *ellipse* or a *circle.*

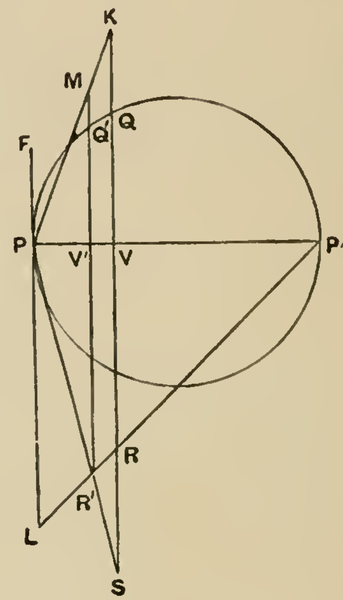


Let PF be parallel to the ordinates to PP', and, if possible, let PK fall between PF and the curve. Draw KV parallel to the ordinates, meeting the curve in Q, and draw VR perpendicular to PV. Join $P'L$ and let it (produced if necessary) meet VR in R.

Then $QV^2 = PV \cdot VR$, so that $KV^2 > PV \cdot VR$.

Take a point S on VR produced such that $KV^2 = PV \cdot VS$. Join PS and let it meet $P'R$ in R'. Draw $R'V'$ parallel to PL meeting PV in V', and through V' draw $V'Q'M$ parallel to QV, meeting the curve in Q' and PK in M.





Now $$KV^2 = PV \cdot VS,$$

$$\therefore\ VS : KV = KV : PV,$$

so that $$VS : PV = KV^2 : PV^2.$$

Hence, by parallels,

$$V'R' : PV' = MV'^2 : PV'^2,$$

or MV' is a mean proportional between PV', $V'R'$,

i.e. $$MV'^2 = PV' \cdot V'R'$$

$$= Q'V'^2, \text{ by the property of the conic.}$$

$$\therefore\ MV' = Q'V'.$$

Thus PK cuts the curve in Q', and therefore does not fall outside it: which is contrary to the hypothesis.

Hence no straight line can fall between PF and the curve.