THE CONE.

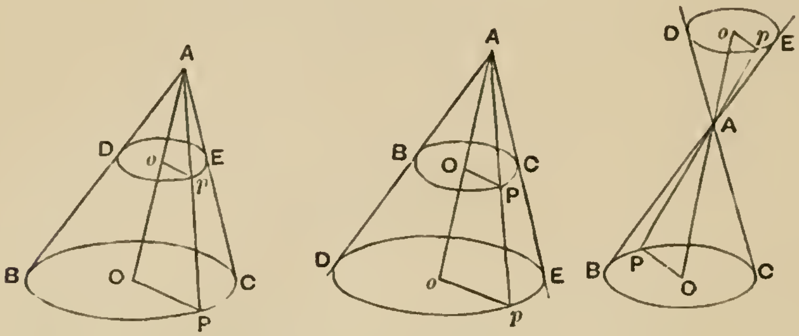
If a straight line indefinite in length, and passing always through a fixed point, be made to move round the circumference of a circle which is not in the same plane with the point, so as to pass successively through every point of that circumference, the moving straight line will trace out the surface of a **double cone**, or two similar cones lying in opposite directions and meeting in the fixed point, which is the **apex** of each cone.

The circle about which the straight line moves is called the **base** of the cone lying between the said circle and the fixed point, and the **axis** is defined as the straight line drawn from the fixed point or the apex to the centre of the circle forming the base.

The cone so described is a **scalene** or **oblique** cone except in the particular case where the axis is perpendicular to the base. In this latter case the cone is a **right** cone.

If a cone be cut by a plane passing through the apex, the resulting section is a triangle, two sides being straight lines lying on the surface of the cone and the third side being the straight line which is the intersection of the cutting plane and the plane of the base.

Let there be a cone whose apex is A and whose base is the circle BC, and let O be the centre of the circle, so that AO is the axis of the cone. Suppose now that the cone is cut by any plane parallel to the plane of the base BC, as DE, and let the axis AO meet the plane DE in o. Let p be any point on the intersection of the plane DE and the surface of the cone. Join Ap and produce it to meet the circumference of the circle BC in P. Join OP, op.



Then, since the plane passing through the straight lines AO, AP cuts the two parallel planes BC, DE in the straight lines OP, op respectively, OP, op are parallel.

$$\therefore\ op : OP = Ao : AO.$$

And, BPC being a circle, OP remains constant for all positions of p on the curve DpE, and the ratio $Ao : AO$ is also constant.

Therefore op is constant for all points on the section of the surface by the plane DE. In other words, that section is a circle.

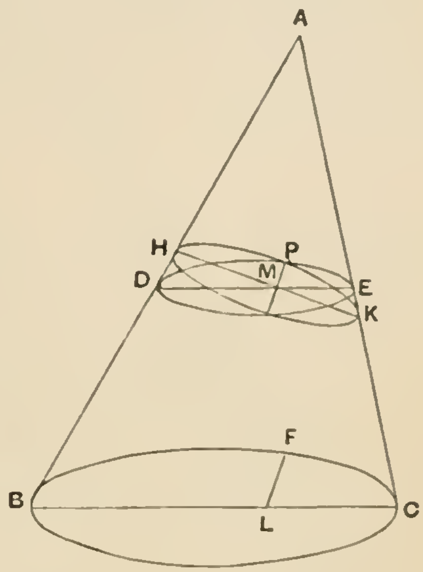
Hence *all sections of the cone which are parallel to the circular base are circles.* [I. 4.]*

Next, let the cone be cut by a plane passing through the axis and perpendicular to the plane of the base BC, and let the section be the triangle ABC. Conceive another plane HK drawn at right angles to the plane of the triangle ABC and cutting off from it the triangle AHK such that AHK is similar to the triangle ABC but lies in the contrary sense, i.e. such that the angle AKH is equal to the angle ABC. Then the section of the cone by the plane HK is called a **subcontrary** section (*ὑπεναντία τομή*).

* The references in this form, here and throughout the book, are to the original propositions of Apollonius.

Let P be any point on the intersection of the plane HK with the surface, and F any point on the circumference of the circle BC. Draw PM, FL each perpendicular to the plane of the triangle ABC, meeting the straight lines HK, BC respectively in M, L. Then PM, FL are parallel.

Draw through M the straight line DE parallel to BC, and it follows that the plane through DME, PM is parallel to the base BC of the cone.



Thus the section DPE is a circle, and $DM \,.\, ME = PM^2$.

But, since DE is parallel to BC, the angle ADE is equal to the angle ABC which is by hypothesis equal to the angle AKH.

Therefore in the triangles HDM, EKM the angles HDM, EKM are equal, as also are the vertical angles at M.

Therefore the triangles HDM, EKM are similar.

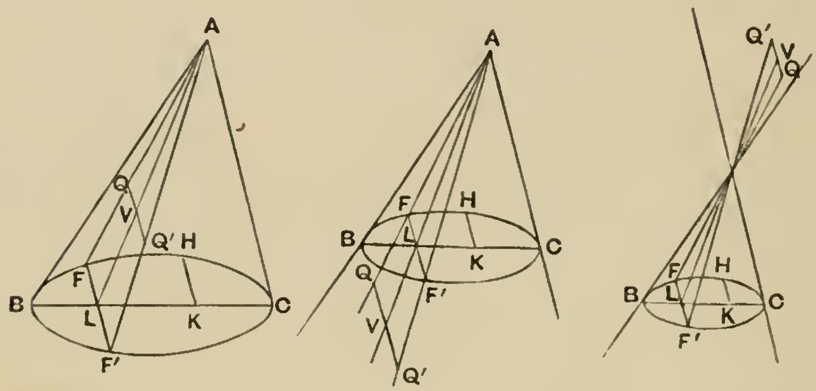
Hence $$HM : MD = EM : MK.$$

$$\therefore\; HM \,.\, MK = DM \,.\, ME = PM^2.$$

And P is *any* point on the intersection of the plane HK with the surface. Therefore the section made by the plane HK is a circle.

Thus *there are two series of circular sections of an oblique cone, one series being parallel to the base, and the other consisting of the sections subcontrary to the first series.* [I. 5.]

Suppose a cone to be cut by any plane through the axis making the triangular section ABC, so that BC is a diameter of the circular base. Let H be any point on the circumference of the base, let HK be perpendicular to the diameter BC, and let a parallel to HK be drawn from any point Q on the surface of the cone but not lying in the plane of the axial triangle. Further, let AQ be joined and produced, if necessary, to meet the circumference of the base in F, and let FLF' be the chord perpendicular to BC. Join AL, AF'. Then the straight line through Q parallel to HK is also parallel to FLF'; it follows therefore that the parallel through Q will meet both AL and AF'. And AL is in the plane of the axial triangle ABC. Therefore the parallel through Q will meet both the plane of the axial triangle and the other side of the surface of the cone, since AF' lies on the cone.



Let the points of intersection be V, Q' respectively.

Then $\qquad QV : VQ' = FL : LF'$, and $FL = LF'$.

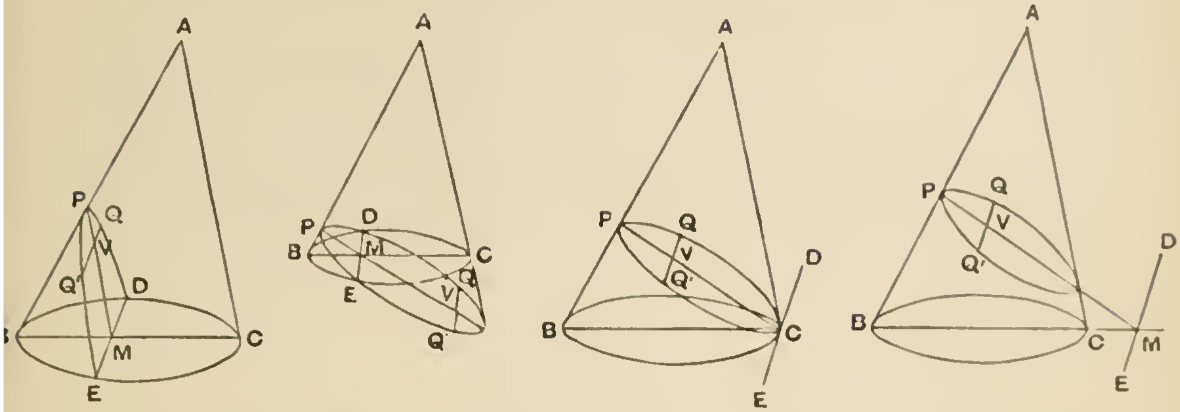
$$\therefore QV = VQ',$$

or QQ' is bisected by the plane of the axial triangle. [I. 6.]

Again, let the cone be cut by another plane not passing through the apex but intersecting the plane of the base in a straight line DME perpendicular to BC, the base of any axial triangle, and let the resulting section of the surface of the cone be DPE, the point P lying on either of the sides AB, AC of the axial triangle. The plane of the section will then cut the plane of the axial triangle in the straight line PM joining P to the middle point of DE.

Now let Q be any point on the curve of section, and through Q draw a straight line parallel to DE.

Then this parallel will, if produced to meet the other side of the surface in Q', meet, and be bisected by, the axial triangle. But it lies also in the plane of the section DPE; it will therefore meet, and be bisected by, PM.



Therefore PM bisects any chord of the section which is parallel to DE.

Now a straight line bisecting each of a series of parallel chords of a section of a cone is called a **diameter**.

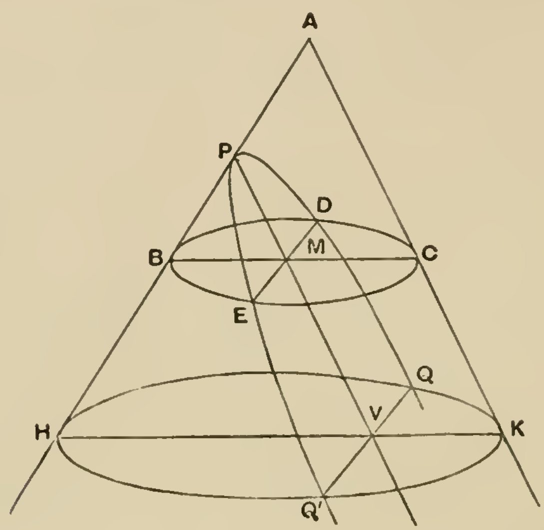
Hence, *if a cone be cut by a plane which intersects the circular base in a straight line perpendicular to the base of any axial triangle, the intersection of the cutting plane and the plane of the axial triangle will be a diameter of the resulting section of the cone.* [I. 7.]

If the cone be a *right* cone it is clear that the diameter so found will, for all sections, be at right angles to the chords which it bisects.

If the cone be *oblique*, the angle between the diameter so found and the parallel chords which it bisects will in general not be a right angle, but will be a right angle in the particular case only where the plane of the axial triangle ABC is at right angles to the plane of the base.

Again, if PM be the diameter of a section made by a plane cutting the circular base in the straight line DME perpendicular to BC, and if PM be in such a direction that it does not meet AC though produced to infinity, i.e. if PM be either parallel to AC, or makes with PB an angle less than the angle BAC and therefore meets CA produced beyond the apex of the cone, the section made by the said plane extends to infinity.

For, if we take any point V on PM produced and draw through it HK parallel to BC, and QQ' parallel to DE, the plane through HK, QQ' is parallel to that through DE, BC, i.e. to the base. Therefore the section $HQKQ'$ is a circle. And D, E, Q, Q' are all on the surface of the cone and are also on the cutting plane. Therefore the section DPE extends to the circle HQK, and in like manner to the circular section through any point on PM produced, and therefore to any distance from P. [I. 8.]

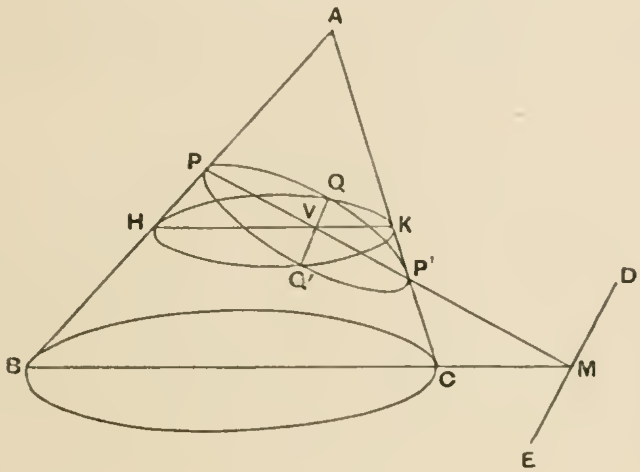


[It is also clear that $DM^2 = BM \,.\, MC$, and $QV^2 = HV \,.\, VK$; and $HV \,.\, VK$ becomes greater as V is taken more distant from P. For, in the case where PM is parallel to AC, VK remains constant while HV increases; and in the case where the diameter PM meets CA produced beyond the apex of the cone, both HV, VK increase together as V moves away from P. Thus QV increases indefinitely as the section extends to infinity.]

If on the other hand PM meets AC, the section does not extend to infinity. In that case the section will be a circle if its plane is parallel to the base or subcontrary. But, if the section is neither parallel to the base nor subcontrary, it will not be a circle. [I. 9.]

For let the plane of the section meet the plane of the base in DME, a straight line perpendicular to BC, a diameter of the circular base. Take the axial triangle through BC meeting the plane of section in the straight line PP'. Then P, P', M are all points in the plane of the axial triangle and in the plane of section. Therefore $PP'M$ is a straight line.

If possible, let the section PP' be a circle. Take any point Q on it and draw QQ' parallel to DME. Then if QQ' meets the axial triangle in V, $QV = VQ'$. Therefore PP' is the diameter of the supposed circle.



Let $HQKQ'$ be the circular section through QQ' parallel to the base.

Then, from the circles, $QV^2 = HV \,.\, VK$,

$$QV^2 = PV \,.\, VP'.$$

$$\therefore HV \,.\, VK = PV \,.\, VP',$$

so that $$HV : VP = P'V : VK.$$

$\therefore$ the triangles VPH, VKP' are similar, and

$$\angle PHV = \angle KP'V;$$

$\therefore \angle KP'V = \angle ABC$, and the section PP' is subcontrary: which contradicts the hypothesis.

$$\therefore PQP' \text{ is not a circle.}$$

It remains to investigate the character of the sections mentioned on the preceding page, viz. (a) those which extend to infinity, (b) those which are finite but are not circles.

Suppose, as usual, that the plane of section cuts the circular base in a straight line DME and that ABC is the axial triangle whose base BC is that diameter of the base of the cone which bisects DME at right angles at the point M. Then, if the plane of the section and the plane of the axial triangle intersect in the straight line PM, PM is a diameter of the section bisecting all chords of the section, as QQ', which are drawn parallel to DE.

If QQ' is so bisected in V, QV is said to be an **ordinate**, or a straight line **drawn ordinate-wise** (τεταγμένως κατηγμένη), to the diameter PM; and the length PV cut off from the diameter by any ordinate QV will be called the **abscissa** of QV.

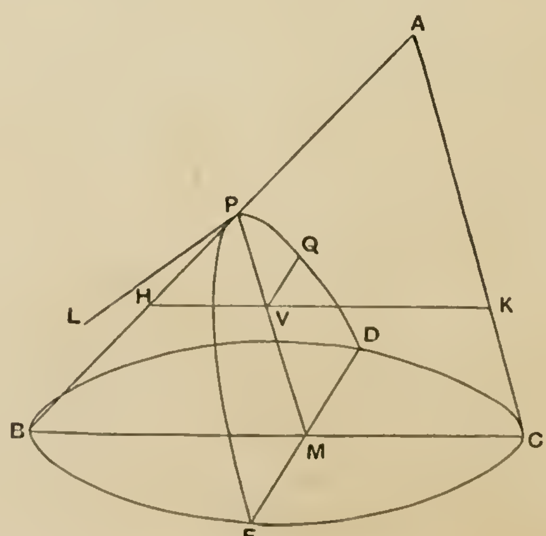
Proposition 1.

[I. 11.]

First let the diameter PM of the section be parallel to one of the sides of the axial triangle as AC, and let QV be any ordinate to the diameter PM. Then, if a straight line PL (supposed to be drawn perpendicular to PM in the plane of the section) be taken of such a length that $PL : PA = BC^2 : BA . AC$, it is to be proved that

$$QV^2 = PL . PV.$$

Let HK be drawn through V parallel to BC. Then, since QV is also parallel to DE, it follows that the plane through H, Q, K is parallel to the base of the cone and therefore



produces a circular section whose diameter is HK. Also QV is at right angles to HK.

$$\therefore\ HV \cdot VK = QV^2.$$

Now, by similar triangles and by parallels,

$$HV : PV = BC : AC$$

and $$VK : PA = BC : BA.$$

$$\therefore\ HV \cdot VK : PV \cdot PA = BC^2 : BA \cdot AC.$$

Hence $$QV^2 : PV \cdot PA = PL : PA$$

$$= PL \cdot PV : PV \cdot PA.$$

$$\therefore\ QV^2 = PL \cdot PV.$$

It follows that the square on any ordinate to the fixed diameter PM is equal to a rectangle applied (παραβάλλειν) to the fixed straight line PL drawn at right angles to PM with altitude equal to the corresponding abscissa PV. Hence the section is called a PARABOLA.

The fixed straight line PL is called the **latus rectum** (ὀρθία) or the **parameter of the ordinates** (παρ' ἣν δύνανται αἱ καταγόμεναι τεταγμένως).

This parameter, corresponding to the diameter PM, will for the future be denoted by the symbol p.

Thus $$QV^2 = p \cdot PV,$$

or $$QV^2 \propto PV.$$

Proposition 2.

[I. 12.]

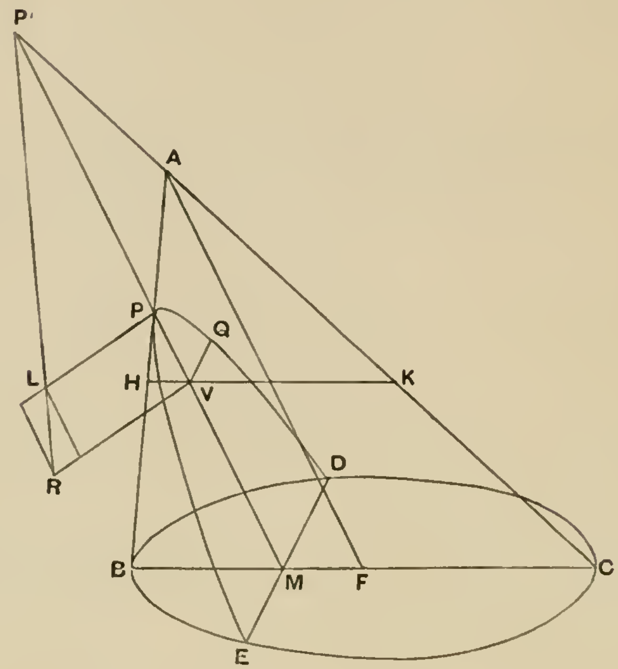
Next let PM not be parallel to AC but let it meet CA produced beyond the apex of the cone in P'. Draw PL at right angles to PM in the plane of the section and of such a length that $PL : PP' = BF \cdot FC : AF^2$, where AF is a straight line through A parallel to PM and meeting BC in F. Then, if VR be drawn parallel to PL and $P'L$ be joined and produced to meet VR in R, it is to be proved that

$$QV^2 = PV \cdot VR.$$

As before, let HK be drawn through V parallel to BC, so that $$QV^2 = HV \cdot VK.$$

Then, by similar triangles,

$$HV : PV = BF : AF,$$
$$VK : P'V = FC : AF.$$



$$\therefore\ HV.VK : PV.P'V = BF.FC : AF^2.$$

Hence $$QV^2 : PV.P'V = PL : PP'$$
$$= VR : P'V$$
$$= PV.VR : PV.P'V.$$
$$\therefore\ QV^2 = PV.VR.$$

It follows that the square on the ordinate is equal to a rectangle whose height is equal to the abscissa and whose base lies along the fixed straight line PL but overlaps (ὑπερβάλλει) it by a length equal to the difference between VR and PL*. Hence the section is called a HYPERBOLA.

* Apollonius describes the rectangle PR as *applied to the latus rectum but exceeding by a figure similar and similarly situated to that contained by PP' and PL*, i.e. exceeding the rectangle VL by the rectangle LR. Thus, if $QV=y$, $PV=x$, $PL=p$, and $PP'=d$,

$$y^2 = px + \frac{p}{d}.x^2,$$

which is simply the Cartesian equation of the hyperbola referred to oblique axes consisting of a diameter and the tangent at its extremity.

PL is called the **latus rectum** or the **parameter of the ordinates** as before, and PP' is called the **transverse** (ἡ πλαγία). The fuller expression **transverse diameter** (ἡ πλαγία διάμετρος) is also used; and, even more commonly, Apollonius speaks of the diameter and the corresponding parameter together, calling the latter the **latus rectum** (i.e. the *erect side*, ἡ ὀρθία πλευρά), and the former the **transverse side** (ἡ πλαγία πλευρά), of the **figure** (εἶδος) **on,** or **applied to, the diameter** (πρὸς τῇ διαμέτρῳ), i.e. of the rectangle contained by PL, PP' as drawn.

The parameter PL will in future be denoted by p.

[Cor. It follows from the proportion

$$QV^2 : PV . P'V = PL : PP'$$

that, for any fixed diameter PP',

$$QV^2 : PV . P'V \text{ is a constant ratio,}$$

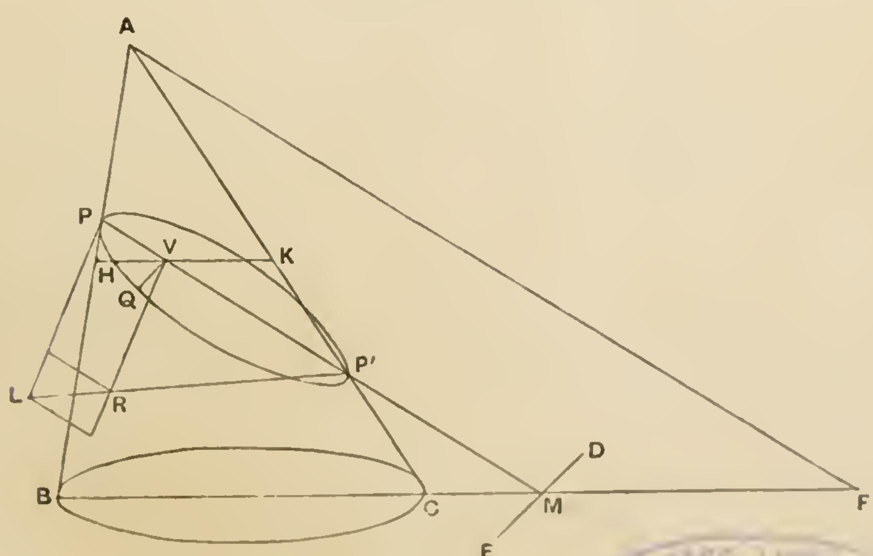
or QV^2 varies as $PV . P'V$.]

Proposition 3.

[I. 13.]

If PM meets AC in P' and BC in M, draw AF parallel to PM meeting BC produced in F, and draw PL at right angles to PM in the plane of the section and of such a length that $PL : PP' = BF . FC : AF^2$. Join $P'L$ and draw VR parallel to PL meeting $P'L$ in R. It will be proved that

$$QV^2 = PV . VR.$$



Draw HK through V parallel to BC. Then, as before,

$$QV^2 = HV . VK.$$

Now, by similar triangles,

$$HV : PV = BF : AF,$$

$$VK : P'V = FC : AF.$$

$$\therefore HV . VK : PV . P'V = BF . FC : AF^2.$$

Hence
$$QV^2 : PV . P'V = PL : PP'$$
$$= VR : P'V$$
$$= PV . VR : PV . P'V.$$
$$\therefore QV^2 = PV . VR.$$

Thus the square on the ordinate is equal to a rectangle whose height is equal to the abscissa and whose base lies along the fixed straight line PL but falls short of it (ἐλλείπει) by a length equal to the difference between VR and PL*. The section is therefore called an Ellipse.

As before, PL is called the **latus rectum**, or the **parameter** of the ordinates to the diameter PP', and PP' itself is called the **transverse** (with or without the addition of **diameter** or **side** of the **figure**, as explained in the last proposition).

PL will henceforth be denoted by p.

[Cor. It follows from the proportion

$$QV^2 : PV . PV' = PL : PP'$$

that, for any fixed diameter PP',

$$QV^2 : PV . P'V \text{ is a constant ratio,}$$

or QV^2 varies as $PV . P'V$.]

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If $QV=y$, $PV=x$, $PL=p$, and $PP'=d$,

$$y^2 = px - \frac{p}{d} . x^2.$$

Thus Apollonius' enunciation simply expresses the Cartesian equation referred to a diameter and the tangent at its extremity as (oblique) axes.