COROLLARY II

Secondly, it follows that, starting from any initial point, if we take any two distances, traversed in any time-intervals whatsoever, these time-intervals bear to one another the same ratio as one of the distances to the mean proportional of the two distances.

Fig. 50

For if we take two distances ST and SY measured from the initial point S, the mean proportional of which is SX, the time of fall through ST is to the time of fall through SY as ST is to SX; or one may say the time of fall through SY is to the time of fall through ST as SY is to SX. Now since it has been shown that the spaces traversed are in the same ratio as the squares of the times; and since, moreover, the ratio of the space SY to the space ST is the square of the ratio SY to SX, it follows that the ratio of the times of fall through SY and ST is the ratio of the respective distances SY and SX.

SCHOLIUM

The above corollary has been proven for the case of vertical fall; but it holds also for planes inclined at any angle; for it is to be assumed that along these planes the velocity increases in the same ratio, that is, in proportion to the time, or, if you prefer, as the series of natural numbers.*

Salv. Here, Sagredo, I should like, if it be not too tedious to Simplicio, to interrupt for a moment the present discussion in order to make some additions on the basis of what has already been proved and of what mechanical principles we have already learned from our Academician. This addition I make for the better establishment on logical and experimental grounds, of the principle which we have above considered; and what is more important, for the purpose of deriving it geometrically, after first demonstrating a single lemma which is fundamental in the science of motion [*impeti*].

* The dialogue which intervenes between this Scholium and the following theorem was elaborated by Viviani, at the suggestion of Galileo. See *National Edition*, viii, 23. [*Trans.*]

Sagr. If the advance which you propose to make is such as will confirm and fully establish these sciences of motion, I will gladly devote to it any length of time. Indeed, I shall not only

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be glad to have you proceed, but I beg of you at once to satisfy the curiosity which you have awakened in me concerning your proposition; and I think that Simplicio is of the same mind.

Simp. Quite right.

Salv. Since then I have your permission, let us first of all consider this notable fact, that the momenta or speeds [*i momenti o le velocità*] of one and the same moving body vary with the inclination of the plane.

The speed reaches a maximum along a vertical direction, and for other directions diminishes as the plane diverges from the vertical. Therefore the impetus, ability, energy, [*l'impeto, il talento, l'energia*] or, one might say, the momentum [*il momento*] of descent of the moving body is diminished by the plane upon which it is supported and along which it rolls.

For the sake of greater clearness erect the line AB perpendicular to the horizontal AC; next draw AD, AE, AF, etc., at different inclinations to the horizontal. Then I say that all the momentum of the falling body is along the vertical and is a maximum when it falls in that direction; the momentum is less along DA and still less along EA, and even less yet along the more inclined plane FA. Finally on the horizontal plane the momentum vanishes altogether; the body finds itself in a condition of indifference as to motion or rest; has no inherent tendency to move in any direction, and offers no resistance to being set in motion. For just as a heavy body or system of bodies cannot of itself move upwards, or recede from the common center [*comun centro*] toward which all heavy things tend, so it is impossible for any body of its own accord to assume any motion other than one which carries it nearer to the aforesaid common center. Hence, along the horizontal, by which we understand a surface, every point of which is equidistant from this same common center, the body will have no momentum whatever.

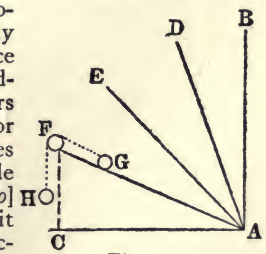

Fig. 51

This

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This change of momentum being clear, it is here necessary for me to explain something which our Academician wrote when in Padua, embodying it in a treatise on mechanics prepared solely for the use of his students, and proving it at length and conclusively when considering the origin and nature of that marvellous machine, the screw. What he proved is the manner in which the momentum [*impeto*] varies with the inclination of the plane, as for instance that of the plane FA, one end of which is elevated through a vertical distance FC. This direction FC is that along which the momentum of a heavy body becomes a maximum; let us discover what ratio this momentum bears to that of the same body moving along the inclined plane FA. This ratio, I say, is the inverse of that of the aforesaid lengths. Such is the lemma preceding the theorem which I hope to demonstrate a little later.

It is clear that the impelling force [*impeto*] acting on a body in descent is equal to the resistance or least force [*resistenza o forza minima*] sufficient to hold it at rest. In order to measure this force and resistance [*forza e resistenza*] I propose to use the weight of another body. Let us place upon the plane FA a body G connected to the weight H by means of a cord passing over the point F; then the body H will ascend or descend, along the perpendicular, the same distance which the body G ascends or descends along the inclined plane FA; but this distance will not be equal to the rise or fall of G along the vertical in which direction alone G, as other bodies, exerts its force [*resistenza*]. This is clear. For if we consider the motion of the body G, from A to F, in the triangle AFC to be made up of a horizontal component AC and a vertical component CF, and remember that this body experiences no resistance to motion along the horizontal (because by such a

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motion the body neither gains nor loses distance from the common center of heavy things) it follows that resistance is met only in consequence of the body rising through the vertical distance CF. Since then the body G in moving from A to F offers resistance only in so far as it rises through the vertical distance CF, while the other body H must fall vertically through the entire distance FA, and since this ratio is maintained whether the motion be large or small, the two bodies being inextensibly connected, we are able to assert positively that, in case of equilibrium (bodies at rest) the momenta,

momenta, the velocities, or their tendency to motion [*propensioni al moto*], i. e., the spaces which would be traversed by them in equal times, must be in the inverse ratio to their weights. This is what has been demonstrated in every case of mechanical motion.* So that, in order to hold the weight G at rest, one must give H a weight smaller in the same ratio as the distance CF is smaller than FA. If we do this, FA:FC=weight G:weight H; then equilibrium will occur, that is, the weights H and G will have the same impelling forces [*momenti eguali*], and the two bodies will come to rest.

And since we are agreed that the impetus, energy, momentum or tendency to motion of a moving body is as great as the force or least resistance [*forza o resistenza minima*] sufficient to stop it, and since we have found that the weight H is capable of preventing motion in the weight G, it follows that the less weight H whose entire force [*momento totale*] is along the perpendicular, FC, will be an exact measure of the component of force [*momento parziale*] which the larger weight G exerts along the plane FA. But the measure of the total force [*total momento*] on the body G is its own weight, since to prevent its fall it is only necessary to balance it with an equal weight, provided this second weight be free to move vertically; therefore the component of the force [*momento parziale*] on G along the inclined plane FA will bear to the maximum and total force on this same body G along the perpendicular FC the same ratio as the weight H to the weight G. This ratio is, by construction, the same which the height, FC, of the inclined plane bears to the length FA. We have here the lemma which I proposed to demonstrate and which, as you will see, has been assumed by our Author in the second part of the sixth proposition of the present treatise.

SAGR. From what you have shown thus far, it appears to me that one might infer, arguing *ex aequali con la proportione perturbata*, that the tendencies [*momenti*] of one and the same body to move along planes differently inclined, but having the same vertical height, as FA and FI, are to each other inversely as the lengths of the planes.

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SALV. Perfectly right. This point established, I pass to the demonstration of the following theorem:

* A near approach to the principle of virtual work enunciated by John Bernoulli in 1717. [*Trans.*]

If a body falls freely along smooth planes inclined at any angle whatsoever, but of the same height, the speeds with which it reaches the bottom are the same.

First we must recall the fact that on a plane of any inclination whatever a body starting from rest gains speed or momentum [*la quantità dell'impeto*] in direct proportion to the time, in agreement with the definition of naturally accelerated motion given by the Author. Hence, as he has shown in the preceding proposition, the distances traversed are proportional to the squares of the times and therefore to the squares of the speeds. The speed relations are here the same as in the motion first studied [i. e., *vertical motion*], since in each case the gain of speed is proportional to the time.

Let AB be an inclined plane whose height above the level BC is AC. As we have seen above the force impelling [*l'impeto*] a body to fall along the vertical AC is to the force which drives the same body along the inclined plane AB as AB is to AC. On the incline AB, lay off AD a third proportional to AB and AC; then the force producing motion along AC is to that along AB (i. e., along AD) as the length AC is to the length AD. And therefore the body will traverse the space AD, along the incline AB, in the same time which it would occupy in falling the vertical distance AC, (since the forces [*momenti*] are in the same ratio as these distances); also the speed at C is to the speed at D as the distance AC is to the distance AD. But, according to the definition of accelerated motion, the speed at B is to the speed of the same body at D as the time required to traverse AB is to the time required for AD; and, according to the last corollary of the second proposition, the time of passing through the distance AB bears to the time of passing through AD the same ratio as the distance AC (a mean proportional between AB and AD) to AD. Accordingly the two speeds at B and C each bear to the speed at D the same ratio, namely, that of the distances AC and AD; hence they are equal. This is the theorem which I set out to prove.

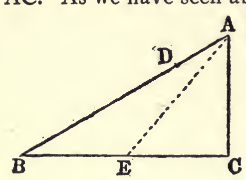

Fig. 52

From the above we are better able to demonstrate the following third proposition of the Author in which he employs the following principle, namely, the time required to traverse an inclined plane is

is to that required to fall through the vertical height of the plane in the same ratio as the length of the plane to its height.

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For, according to the second corollary of the second proposition, if BA represents the time required to pass over the distance BA, the time required to pass the distance AD will be a mean proportional between these two distances and will be represented by the line AC; but if AC represents the time needed to traverse AD it will also represent the time required to fall through the distance AC, since the distances AC and AD are traversed in equal times; consequently if AB represents the time required for AB then AC will represent the time required for AC. Hence the times required to traverse AB and AC are to each other as the distances AB and AC.

In like manner it can be shown that the time required to fall through AC is to the time required for any other incline AE as the length AC is to the length AE; therefore, *ex aequali*, the time of fall along the incline AB is to that along AE as the distance AB is to the distance AE, etc.*

One might by application of this same theorem, as Sagredo will readily see, immediately demonstrate the sixth proposition of the Author; but let us here end this digression which Sagredo has perhaps found rather tedious, though I consider it quite important for the theory of motion.

SAGR. On the contrary it has given me great satisfaction, and indeed I find it necessary for a complete grasp of this principle.

SALV. I will now resume the reading of the text.

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THEOREM III, PROPOSITION III

If one and the same body, starting from rest, falls along an inclined plane and also along a vertical, each having the same height, the times of descent will be to each other as the lengths of the inclined plane and the vertical.

Let AC be the inclined plane and AB the perpendicular, each having the same vertical height above the horizontal, namely, BA; then I say, the time of descent of one and the same body

* Putting this argument in a modern and evident notation, one has $AC = \frac{1}{2}\,gt_c^2$ and $AD = \frac{1}{2}\,\frac{AC}{AB}\,gt_d^2$ If now $\overline{AC}^2 = AB.\,AD$, it follows at once that $t_d = t_c$. [*Trans.*]

Q. D. E.

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along the plane AC bears a ratio to the time of fall along the perpendicular AB, which is the same as the ratio of the length AC to the length AB. Let DG, EI and LF be any lines parallel to the horizontal CB; then it follows from what has preceded that a body starting from A will acquire the same speed at the point G as at D, since in each case the vertical fall is the same; in like manner the speeds at I and E will be the same; so also those at L and F. And in general the speeds at the two extremities of any parallel drawn from any point on AB to the corresponding point on AC will be equal.

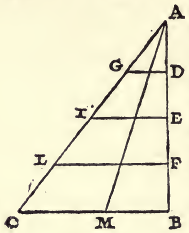

Fig. 53

Thus the two distances AC and AB are traversed at the same speed. But it has already been proved

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that if two distances are traversed by a body moving with equal speeds, then the ratio of the times of descent will be the ratio of the distances themselves; therefore, the time of descent along AC is to that along AB as the length of the plane AC is to the vertical distance AB. Q. E. D.

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SAGR. It seems to me that the above could have been proved clearly and briefly on the basis of a proposition already demonstrated, namely, that the distance traversed in the case of accelerated motion along AC or AB is the same as that covered

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by a uniform speed whose value is one-half the maximum speed, CB; the two distances AC and AB having been traversed at the same uniform speed it is evident, from Proposition I, that the times of descent will be to each other as the distances.

COROLLARY

Hence we may infer that the times of descent along planes having different inclinations, but the same vertical height stand to

to one another in the same ratio as the lengths of the planes. For consider any plane AM extending from A to the horizontal CB; then it may be demonstrated in the same manner that the time of descent along AM is to the time along AB as the distance AM is to AB; but since the time along AB is to that along AC as the length AB is to the length AC, it follows, *ex æquali*, that as AM is to AC so is the time along AM to the time along AC.

THEOREM IV, PROPOSITION IV

The times of descent along planes of the same length but of different inclinations are to each other in the inverse ratio of the square roots of their heights

From a single point B draw the planes BA and BC, having the same length but different inclinations; let AE and CD be horizontal lines drawn to meet the perpendicular BD; and

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let BE represent the height of the plane AB, and BD the height of BC; also let BI be a mean proportional to BD and BE; then the ratio of BD to BI is equal to the square root of the ratio of BD to BE. Now, I say, the ratio of the times of descent along BA and BC is the ratio of BD to BI; so that the time of descent along BA is related to the height of the other plane BC, namely BD as the time along BC is related to the height BI. Now it must be proved that the time of descent along BA is to that along BC as the length BD is to the length BI.

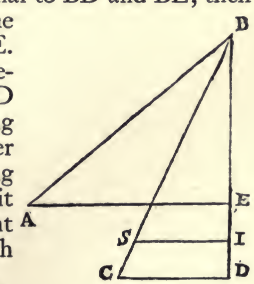

Fig. 54

Draw IS parallel to DC; and since it has been shown that the time of fall along BA is to that along the vertical BE as BA is to BE; and also that the time along BE is to that along BD as BE is to BI; and likewise that the time along BD is to that along BC as BD is to BC, or as BI to BS; it follows, *ex æquali*, that the time along BA is to that along BC as BA to BS, or BC to BS. However, BC is to BS as BD is to BI; hence follows our proposition.

Theorem V, Proposition V

The times of descent along planes of different length, slope and height bear to one another a ratio which is equal to the product of the ratio of the lengths by the square root of the inverse ratio of their heights.

Draw the planes AB and AC, having different inclinations, lengths, and heights. My theorem then is that the ratio of the time of descent along AC to that along AB is equal to the product of the ratio of AC to AB by the square root of the inverse ratio of their heights.

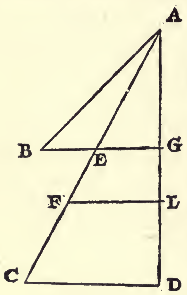

Fig. 55

For let AD be a perpendicular to which are drawn the horizontal lines BG and CD; also let AL be a mean proportional to the heights AG and AD; from the point L draw a horizontal line meeting AC in F; accordingly AF will be a mean proportional between AC and AE. Now since the time of descent along AC is to that along AE as the length AF is to AE; and since the time along AE is to that along AB as AE is to AB, it is clear that the time along AC is to that along AB as AF is to AB.

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Thus it remains to be shown that the ratio of AF to AB is equal to the product of the ratio of AC to AB by the ratio of AG to AL, which is the inverse ratio of the square roots of the heights DA and GA. Now it is evident that, if we consider the line AC in connection with AF and AB, the ratio of AF to AC is the same as that of AL to AD, or AG to AL which is the square root of the ratio of the heights AG and AD; but the ratio of AC to AB is the ratio of the lengths themselves. Hence follows the theorem.

Theorem VI, Proposition VI

If from the highest or lowest point in a vertical circle there be drawn any inclined planes meeting the circumference the times

times of descent along these chords are each equal to the other.

On the horizontal line GH construct a vertical circle. From its lowest point—the point of tangency with the horizontal—draw the diameter FA and from the highest point, A, draw inclined planes to B and C, any points whatever on the circumference; then the times of descent along these are equal. Draw BD and CE perpendicular to the diameter; make AI a mean proportional between the heights of the planes, AE and AD; and since the rectangles FA.AE and FA.AD are respectively equal to the squares of AC and AB, while the rectangle FA.AE is to the rectangle FA.AD as AE is to AD, it follows that the square of AC is to the square of AB as the length AE is to the length AD.

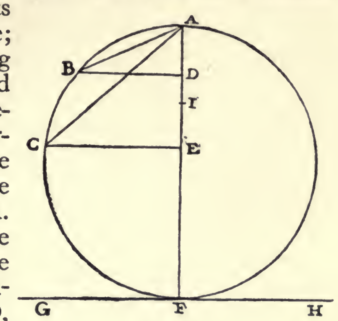

Fig. 56

But since the length AE is to AD as the square of AI is to the square of AD, it follows that the squares on the lines AC and AB are to each other as the squares on the lines AI and AD, and hence also the length AC is to the length AB as AI is to AD. But it has previously been demonstrated that the ratio of the time of descent along AC to that along AB is equal to the product of the two ratios AC to AB and AD to AI; but this last ratio is the same as that of AB to AC. Therefore the ratio of the time of descent along AC to that along AB is the product of the two ratios, AC to AB and AB to AC. The ratio of these times is therefore unity. Hence follows our proposition.

By use of the principles of mechanics [*ex mechanicis*] one may obtain the same result, namely, that a falling body will require equal times to traverse the distances CA and DA, indicated in the following figure. Lay off BA equal to DA, and let fall the

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perpendiculars BE and DF; it follows from the principles of mechanics

mechanics that the component of the momentum [*momentum ponderis*] acting along the inclined plane ABC is to the total momentum [i. e., the momentum of the body falling freely] as BE is to BA; in like manner the momentum along the plane AD is to its total momentum [i. e., the momentum of the body falling freely] as DF is to DA, or to BA. Therefore the momentum of this same weight along the plane DA is to that along the plane ABC as the length DF is to the length BE; for this reason, this same weight will in equal times according to the second proposition of the first book, traverse spaces along the planes CA and DA which are to each other as the lengths BE and DF. But it can be shown that CA is to DA as BE is to DF. Hence the falling body will traverse the two paths CA and DA in equal times.

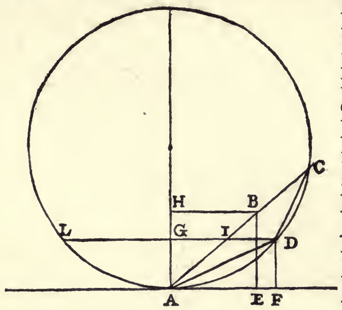

Fig. 57

Moreover the fact that CA is to DA as BE is to DF may be demonstrated as follows: Join C and D; through D, draw the line DGL parallel to AF and cutting the line AC in I; through B draw the line BH, also parallel to AF. Then the angle ADI will be equal to the angle DCA, since they subtend equal arcs LA and DA, and since the angle DAC is common, the sides of the triangles, CAD and DAI, about the common angle will be proportional to each other; accordingly as CA is to DA so is DA to IA, that is as BA is to IA, or as HA is to GA, that is as BE is to DF. E. D.

The same proposition may be more easily demonstrated as follows: On the horizontal line AB draw a circle whose diameter DC is vertical. From the upper end of this diameter draw any inclined plane, DF, extending to meet the circumference; then, I say, a body will occupy the same time in falling along the plane DF as along the diameter DC. For draw FG parallel to

to AB and perpendicular to DC; join FC; and since the time of fall along DC is to that along DG as the mean proportional [223] between CD and GD is to GD itself; and since also DF is a mean proportional between DC and DG, the angle DFC inscribed in a semicircle being a right-angle, and FG being perpendicular to DC, it follows that the time of fall along DC is to that along DG as the length FD is to GD. But it has already been demonstrated that the time of descent along DF is to that along DG as the length DF is to DG; hence the times of descent along DF and DC each bear to the time of fall along DG the same ratio; consequently they are equal.

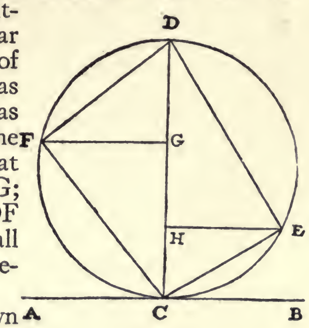

Fig. 58

In like manner it may be shown that if one draws the chord CE from the lower end of the diameter, also the line EH parallel to the horizon, and joins the points E and D, the time of descent along EC, will be the same as that along the diameter, DC.

COROLLARY I

From this it follows that the times of descent along all chords drawn through either C or D are equal one to another.

COROLLARY II

It also follows that, if from any one point there be drawn a vertical line and an inclined one along which the time of descent is the same, the inclined line will be a chord of a semicircle of which the vertical line is the diameter.

COROLLARY III

Moreover the times of descent along inclined planes will be equal when the vertical heights of equal lengths of these planes are

are to each other as the lengths of the planes themselves; thus it is clear that the times of descent along CA and DA, in the figure just before the last, are equal, provided the vertical height of AB (AB being equal to AD), namely, BE, is to the vertical height DF as CA is to DA.

SAGR. Please allow me to interrupt the lecture for a moment in order that I may clear up an idea which just occurs to me; one which, if it involve no fallacy, suggests at least a freakish and

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interesting circumstance, such as often occurs in nature and in the realm of necessary consequences.

If, from any point fixed in a horizontal plane, straight lines be drawn extending indefinitely in all directions, and if we imagine a point to move along each of these lines with constant speed, all starting from the fixed point at the same instant and moving with equal speeds, then it is clear that all of these moving points will lie upon the circumference of a circle which grows larger and larger, always having the aforesaid fixed point as its center; this circle spreads out in precisely the same manner as the little waves do in the case of a pebble allowed to drop into quiet water, where the impact of the stone starts the motion in all directions, while the point of impact remains the center of these ever-expanding circular waves. But imagine a vertical plane from the highest point of which are drawn lines inclined at every angle and extending indefinitely; imagine also that heavy particles descend along these lines each with a naturally accelerated motion and each with a speed appropriate to the inclination of its line. If these moving particles are always visible, what will be the locus of their positions at any instant? Now the answer to this question surprises me, for I am led by the preceding theorems to believe that these particles will always lie upon the circumference of a single circle, ever increasing in size as the particles recede farther and farther from the point at which their motion began. To be more definite, let A be the fixed point from which are drawn the lines AF and AH inclined at any angle whatsoever. On the perpendicular AB take any two points C and D about which, as centers, circles are described passing

passing through the point A, and cutting the inclined lines at the points F, H, B, E, G, I. From the preceding theorems it is clear that, if particles start, at the same instant, from A and descend along these lines, when one is at E another will be at G and another at I; at a later instant they will be found simultaneously at F, H and B; these, and indeed an infinite number of other particles

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travelling along an infinite number of different slopes will at successive instants always lie upon a single ever-expanding circle. The two kinds of motion occurring in nature give rise therefore to two infinite series of circles, at once resembling and differing from each other; the one takes its rise in the center of an infinite number of concentric circles; the other has its origin in the contact, at their highest points, of an infinite number of eccentric circles; the former are produced by motions which are equal and uniform; the latter by motions which are neither uniform nor equal among themselves, but which vary from one to another according to the slope.

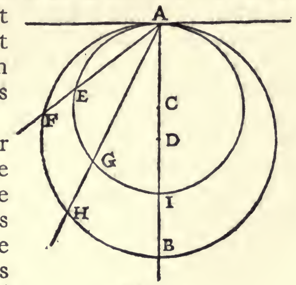

Fig. 59

Further, if from the two points chosen as origins of motion, we draw lines not only along horizontal and vertical planes but in all directions then just as in the former cases, beginning at a single point ever-expanding circles are produced, so in the latter case an infinite number of spheres are produced about a single point, or rather a single sphere which expands in size without limit; and this in two ways, one with the origin at the center, the other on the surface of the spheres.

SALV. The idea is really beautiful and worthy of the clever mind of Sagredo.

SIMP. As for me, I understand in a general way how the two kinds of natural motions give rise to the circles and spheres; and yet as to the production of circles by accelerated motion and its proof, I am not entirely clear; but the fact that one can take the

the origin of motion either at the inmost center or at the very top of the sphere leads one to think that there may be some great mystery hidden in these true and wonderful results, a mystery related to the creation of the universe (which is said to be spherical in shape), and related also to the seat of the first cause [*prima causa*].

Salv. I have no hesitation in agreeing with you. But profound considerations of this kind belong to a higher science than ours [*a più alte dottrine che le nostre*]. We must be satisfied to belong to that class of less worthy workmen who procure from the quarry the marble out of which, later, the gifted sculptor produces those masterpieces which lay hidden in this rough and shapeless exterior. Now, if you please, let us proceed.

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Theorem VII, Proposition VII

If the heights of two inclined planes are to each other in the same ratio as the squares of their lengths, bodies starting from rest will traverse these planes in equal times.

Take two planes of different lengths and different inclinations, AE and AB, whose heights are AF and AD: let AF be to AD as the square of AE is to the square of AB; then, I say, that a body, starting from rest at A, will traverse the planes AE and AB in equal times. From the vertical line, draw the horizontal parallel lines EF and DB, the latter cutting AE at G. Since FA:DA$=\overline{EA}^2:\overline{BA}^2$, and since FA:DA=EA:GA, it follows that EA : GA $=\overline{EA}^2:\overline{BA}^2$. Hence BA is a mean proportional between EA and GA. Now since the time of descent along AB bears to the time along AG the same ratio which AB bears to AG and since also the time of descent along AG is to the time along AE as AG is to a mean proportional between AG and AE, that is, to AB, it follows, *ex æquali*, that

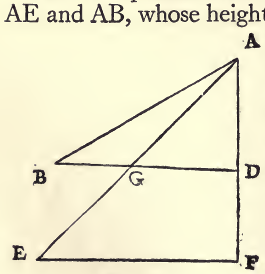

Fig. 60

that the time along AB is to the time along AE as AB is to itself. Therefore the times are equal. Q. E. D.

Theorem VIII, Proposition VIII

The times of descent along all inclined planes which intersect one and the same vertical circle, either at its highest or lowest point, are equal to the time of fall along the vertical diameter; for those planes which fall short of this diameter the times are shorter; for planes which cut this diameter, the times are longer.

Let AB be the vertical diameter of a circle which touches the horizontal plane. It has already been proven that the times of descent along planes drawn from either end, A or B, to the circumference are equal. In order to show that the time of descent [227] along the plane DF which falls short of the diameter is shorter we may draw the plane DB which is both longer and less steeply inclined than DF; whence it follows that the time along DF is less than that along DB and consequently along AB. In like manner, it is shown that the time of descent along CO which cuts the diameter is greater: for it is both longer and less steeply inclined than CB. Hence follows the theorem.

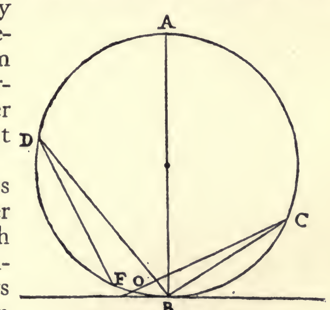

Fig. 61

Theorem IX, Proposition IX

If from any point on a horizontal line two planes, inclined at any angle, are drawn, and if they are cut by a line which makes with them angles alternately equal to the angles between these planes and the horizontal, then the times required to traverse those portions of the plane cut off by the aforesaid line are equal.

Through

Through the point C on the horizontal line X, draw two planes CD and CE inclined at any angle whatever: at any point in the line CD lay off the angle CDF equal to the angle XCE; let the line DF cut CE at F so that the angles CDF and CFD are alternately equal to XCE and LCD; then, I say, the times of descent over CD and CF are equal. Now since the angle CDF is equal to the angle XCE by construction, it is evident that the angle CFD must be equal to the angle DCL. For if the common angle DCF be subtracted from the three angles of the triangle CDF, together equal to two right angles, (to which are also equal all the angles which can be described about the point C on the lower side of the line LX) there remain in the triangle two angles, CDF and CFD, equal to the two angles XCE and LCD; but, by hypothesis, the angles CDF and XCE are equal; hence the remaining angle CFD is equal to the remainder DCL. Take CE equal to CD; from the points D and E draw DA and EB perpendicular to the horizontal line XL; and from the point C draw CG perpendicular to DF. Now since the angle CDG is equal to the angle ECB and since DGC and CBE are right angles, it follows that the triangles CDG and CBE are equiangular; consequently DC:CG=CE:EB. But DC is equal to CE, and therefore CG is equal to EB. Since also the angles at C and at A, in the triangle DAC, are equal to the angles at F and G in the triangle CGF, we have CD:DA=FC:CG and, *permutando*, DC:CF=DA:CG=DA:BE. Thus the ratio of the heights of the equal planes CD and CE is the same as the ratio of the lengths DC and CF. Therefore, by

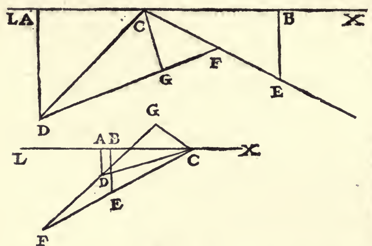

Fig. 62

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Corollary I of Prop. VI, the times of descent along these planes will be equal. Q. E. D.

An alternative proof is the following: Draw FS perpendicular to

to the horizontal line AS. Then, since the triangle CSF is similar to the triangle DGC, we have SF:FC=GC:CD; and since the triangle CFG is similar to the triangle DCA, we have FC:CG=CD:DA. Hence, *ex æquali*, SF:CG=CG:DA. Therefore CG is a mean proportional between SF and DA, while DA:SF = $\overline{DA}^2:\overline{CG}^2$. Again since the triangle ACD is similar to the triangle CGF, we have DA:DC=GC:CF and, *permutando*, DA:CG = DC:CF: also $\overline{DA}^2:\overline{CG}^2=\overline{DC}^2:\overline{CF}^2$. But it has been shown that $\overline{DA}^2:\overline{CG}^2=$ DA:SF. Therefore $\overline{DC}^2:\overline{CF}^2=$DA:FS. Hence from the above Prop. VII, since the heights DA and FS of the planes CD and CF are to each other as the squares of the lengths of the planes, it follows that the times of descent along these planes will be equal.

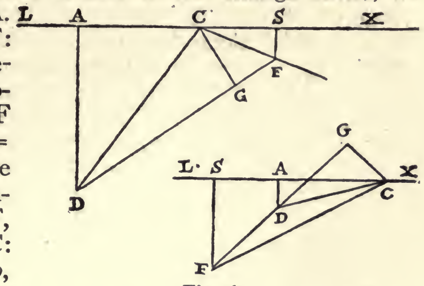

Fig. 63

Theorem X, Proposition X

The times of descent along inclined planes of the same height, but of different slope, are to each other as the lengths of these planes; and this is true whether the motion starts from rest or whether it is preceded by a fall from a constant height.

Let the paths of descent be along ABC and ABD to the horizontal plane DC so that the falls along BD and BC are preceded by the fall along AB; then, I say, that the time of descent along BD is to the time of descent along BC as the length BD is to BC. Draw the horizontal line AF and extend DB until it cuts this

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line at F; let FE be a mean proportional between DF and FB; draw EO parallel to DC; then AO will be a mean proportional between CA and AB. If now we represent the time of fall along AB

AB by the length AB, then the time of descent along FB will be represented by the distance FB; so also the time of fall through the entire distance AC will be represented by the mean proportional AO: and for the entire distance FD by FE. Hence the time of fall along the remainder, BC, will be represented by BO, and that along the remainder, BD, by BE; but since BE:BO=BD:BC, it follows, if we allow the bodies to fall first along AB and FB, or, what is the same thing, along the common stretch AB, that the times of descent along BD and BC will be to each other as the lengths BD and BC.

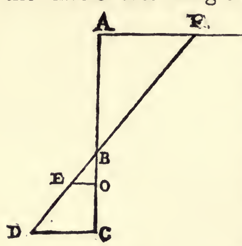

Fig. 64

But we have previously proven that the time of descent, from rest at B, along BD is to the time along BC in the ratio which the length BD bears to BC. Hence the times of descent along different planes of constant height are to each other as the lengths of these planes, whether the motion starts from rest or is preceded by a fall from a constant height. Q. E. D.

Theorem XI, Proposition XI

If a plane be divided into any two parts and if motion along it starts from rest, then the time of descent along the first part is to the time of descent along the remainder as the length of this first part is to the excess of a mean proportional between this first part and the entire length over this first part.

Let the fall take place, from rest at A, through the entire distance AB which is divided at any point C; also let AF be a mean proportional between the entire length BA and the first part AC; then CF will denote the excess of the mean proportional FA over the first part AC. Now, I say, the time of descent along AC will be to the time of subsequent fall through CB as the length AC is to CF. This is evident, because the time along AC is to the time along the entire distance AB as AC is to the mean proportional AF. Therefore,

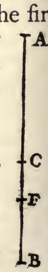

Fig. 65

Therefore, *dividendo*, the time along AC will be to the time along the remainder CB as AC is to CF. If we agree to represent the time along AC by the length AC then the time along CB will be represented by CF. Q. E. D.

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In case the motion is not along the straight line ACB but along the broken line ACD to the horizontal line BD, and if from F we draw the horizontal line FE, it may in like manner be proved that the time along AC is to the time along the inclined line CD as AC is to CE. For the time along AC is to the time along CB as AC is to CF; but it has already been shown that the time along CB, after the fall through the distance AC, is to the time along CD, after descent through the same distance AC, as CB is to CD, or, as CF is to CE; therefore, *ex æquali*, the time along AC will be to the time along CD as the length AC is to the length CE.

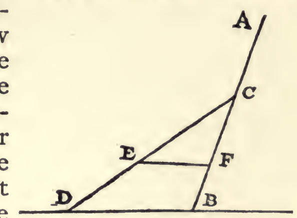

Fig. 66

Theorem XII, Proposition XII

If a vertical plane and any inclined plane are limited by two horizontals, and if we take mean proportionals between the lengths of these planes and those portions of them which lie between their point of intersection and the upper horizontal, then the time of fall along the perpendicular bears to the time required to traverse the upper part of the perpendicular plus the time required to traverse the lower part of the intersecting plane the same ratio which the entire length of the vertical bears to a length which is the sum of the mean proportional on the vertical plus the excess of the entire length of the inclined plane over its mean proportional.

Let AF and CD be two horizontal planes limiting the vertical plane AC and the inclined plane DF; let the two last-mentioned planes intersect at B. Let AR be a mean proportional between the

the entire vertical AC and its upper part AB; and let FS be a mean proportional between FD and its upper part FB. Then, I say, the time of fall along the entire vertical path AC bears to the time of fall along its upper portion AB plus the time of fall along the lower part of the inclined plane, namely, BD, the same ratio which the length AC bears to the mean proportional on the vertical, namely, AR, plus the length SD which is the excess of the entire plane DF over its mean proportional FS.

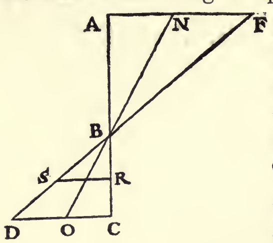

Fig. 67

Join the points R and S giving a horizontal line RS. Now since the time of fall through the entire distance AC is to the time along the portion AB as CA is to the mean proportional AR it follows that, if we agree to represent the time of fall through AC by the distance AC, the time of fall through the distance AB will be represented by AR; and the time of descent through the remainder, BC, will be represented by RC. But, if the time along AC is taken to be equal to the length AC, then the time along FD will be equal to the distance FD; and we may likewise infer that the time of descent along BD, when preceded by a fall along FB or AB, is numerically equal to the distance DS. Therefore

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the time required to fall along the path AC is equal to AR plus RC; while the time of descent along the broken line ABD will be equal to AR plus SD. Q. E. D.

The same thing is true if, in place of a vertical plane, one takes any other plane, as for instance NO; the method of proof is also the same.

Problem I, Proposition XIII

Given a perpendicular line of limited length, it is required to find a plane having a vertical height equal to the given perpendicular and so inclined that a body, having fallen from rest along the perpendicular, will make its descent along

along the inclined plane in the same time which it occupied in falling through the given perpendicular.

Let AB denote the given perpendicular: prolong this line to C making BC equal to AB, and draw the horizontal lines CE and AG. It is required to draw a plane from B to the horizontal line CE such that after a body starting from rest at A has fallen through the distance AB, it will complete its path along this plane in an equal time. Lay off CD equal to BC, and draw the line BD. Construct the line BE equal to the sum of BD and DC; then, I say, BE is the required plane. Prolong EB till it intersects the horizontal AG at G. Let GF be a mean proportional between GE and GB; then $EF:FB=EG:GF$, and $\overline{EF}^2:\overline{FB}^2=\overline{EG}^2:\overline{GF}^2=EG:GB$. But EG is twice GB; hence the square of EF is twice the square of FB; so also is the square of DB twice the square of BC. Consequently $EF:FB=DB:BC$, and *componendo et permutando*, $EB:DB+BC=BF:BC$. But $EB=DB+BC$; hence $BF=BC=BA$. If we agree that the length AB shall represent the time of fall along the line AB, then GB will represent the time of descent along GB, and GF the time along the entire distance GE; therefore BF will represent the time of descent along the difference of these paths, namely, BE, after fall from G or from A. Q. E. F.

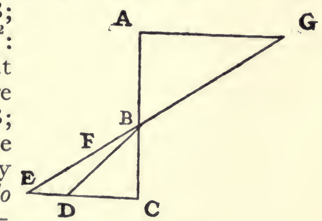

Fig. 68

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Problem II, Proposition XIV

Given an inclined plane and a perpendicular passing through it, to find a length on the upper part of the perpendicular through which a body will fall from rest in the same time which is required to traverse the inclined plane after fall through the vertical distance just determined.

Let AC be the inclined plane and DB the perpendicular. It is required to find on the vertical AD a length which will be traversed

traversed by a body, falling from rest, in the same time which is needed by the same body to traverse the plane AC after the aforesaid fall. Draw the horizontal CB; lay off AE such that $BA+2AC:AC=AC:AE$, and lay off AR such that $BA:AC=EA:AR$. From R draw RX perpendicular to DB; then, I say, X is the point sought. For since $BA+2AC:AC=AC:AE$, it follows, *dividendo*, that $BA+AC:AC=CE:AE$. And since $BA:AC=EA:AR$, we have, *componendo*, $BA+AC:AC=ER:RA$. But $BA+AC:AC=CE:AE$, hence $CE:EA=ER:RA=$ sum of the antecedents: sum of the consequents $=CR:RE$. Thus RE is seen to be a mean proportional between CR and RA. Moreover since it has been assumed that $BA:AC=EA:AR$, and since by similar triangles we have $BA:AC=XA:AR$, it follows that $EA:AR=XA:AR$. Hence EA and XA are equal. But if we agree that the time of fall through RA shall be represented by the length RA, then the time of fall along RC will be represented by the length RE which is a mean proportional between RA and RC; likewise AE will represent the time of descent along AC after descent along RA or along AX. But the time of fall through XA is represented by the length XA, while RA represents the time through RA. But it has been shown that XA and AE are equal. Q. E. F.

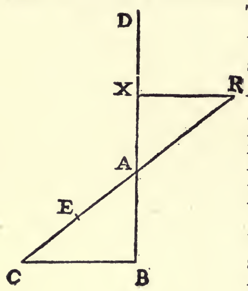

Fig. 69

Problem III, Proposition XV

Given a vertical line and a plane inclined to it, it is required to find a length on the vertical line below its point of intersection which will be traversed in the same time as the inclined plane, each of these motions having been preceded by a fall through the given vertical line.

Let AB represent the vertical line and BC the inclined plane; it is required to find a length on the perpendicular below its point of intersection, which after a fall from A will be traversed in the same

same time which is needed for BC after an identical fall from A. Draw the horizontal AD, intersecting the prolongation of CB at D; let DE be a mean proportional between CD and DB; lay

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off BF equal to BE; also let AG be a third proportional to BA and AF. Then, I say, BG is the distance which a body, after falling through AB, will traverse in the same time which is needed for the plane BC after the same preliminary fall. For if we assume that the time of fall along AB is represented by AB, then the time for DB will be represented by DB. And since DE is a mean proportional between BD and DC, this same DE will represent the time of descent along the entire distance DC while BE will represent the time required for the difference of these paths, namely, BC, provided in each case the fall is from rest at D or at A. In like manner we may infer that BF represents the time of descent through the distance BG after the same preliminary fall; but BF is equal to BE. Hence the problem is solved.

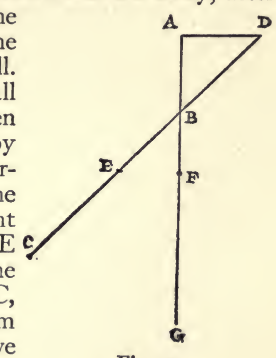

Fig. 70

Theorem XIII, Proposition XVI

If a limited inclined plane and a limited vertical line are drawn from the same point, and if the time required for a body, starting from rest, to traverse each of these is the same, then a body falling from any higher altitude will traverse the inclined plane in less time than is required for the vertical line.

Let EB be the vertical line and CE the inclined plane, both starting from the common point E, and both traversed in equal times by a body starting from rest at E; extend the vertical line upwards to any point A, from which falling bodies are allowed to start. Then, I say that, after the fall through AE, the inclined plane EC will be traversed in less time than the perpendicular

pendicular EB. Join CB, draw the horizontal AD, and prolong CE backwards until it meets the latter in D; let DF be a mean proportional between CD and DE while AG is made a mean proportional between BA and AE. Draw FG and DG; then

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since the times of descent along EC and EB, starting from rest at E, are equal, it follows, according to Corollary II of Proposition VI that the angle at C is a right angle; but the angle at A is also a right angle and the angles at the vertex E are equal; hence the triangles AED and CEB are equiangular and the sides about the equal angles are proportional; hence BE:EC = DE:EA. Consequently the rectangle BE.EA is equal to the rectangle CE.ED; and since the rectangle CD.DE exceeds the rectangle CE.ED by the square of ED, and since the rectangle BA.AE exceeds the rectangle BE.EA by the square of EA, it follows that the excess of the rectangle CD.DE over the rectangle BA.AE, or what is the same thing, the excess of the square of FD over the square of AG, will be equal to the excess of the square of DE over the square of AE, which excess is equal to the square of AD. Therefore $\overline{FD}^2 = \overline{GA}^2 + \overline{AD}^2 = \overline{GD}^2$. Hence DF is equal to DG, and the angle DGF is equal to the angle DFG while the angle EGF is less than the angle EFG, and the opposite side EF is less than the opposite side EG. If now we agree to represent the time of fall through AE by the length AE, then the time along DE will be represented by DE. And since AG is a mean proportional between BA and AE,

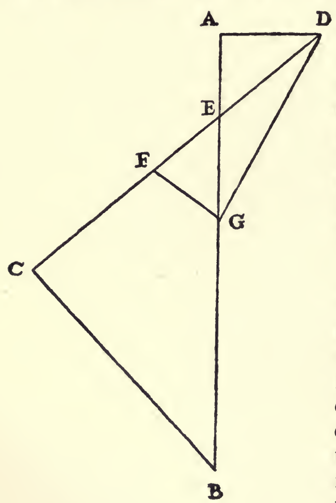

Fig. 71

AE, it follows that AG will represent the time of fall through the total distance AB, and the difference EG will represent the time of fall, from rest at A, through the difference of path EB.

In like manner EF represents the time of descent along EC, starting from rest at D or falling from rest at A. But it has been shown that EF is less than EG; hence follows the theorem.

COROLLARY

From this and the preceding proposition, it is clear that the vertical distance covered by a freely falling body, after a preliminary fall, and during the time-interval required to traverse an inclined plane, is greater than the length of the inclined plane, but less than the distance traversed on the inclined plane during an equal time, without any preliminary fall. For since we have just shown that bodies falling from an elevated point A will traverse the plane EC in Fig. 71 in a shorter time than the vertical EB, it is evident that the distance along EB which will be traversed during a time equal to that of descent along EC will be less than the whole of EB. But now in order to show that this vertical distance is greater than the length of the inclined plane EC, we reproduce Fig. 70 of the preceding theorem in which the vertical length BG is traversed in the same time as BC after a preliminary fall through AB. That BG is greater than BC is shown as follows: since BE and FB are equal

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while BA is less than BD, it follows that FB will bear to BA a greater ratio than EB bears to BD; and, *componendo*, FA will bear to BA a greater ratio than ED to DB; but FA:AB = GF:FB (since AF is a mean proportional between BA and AG) and in like manner ED:BD = CE:EB. Hence GB bears to BF a greater ratio than CB bears to BE; therefore GB is greater than BC.

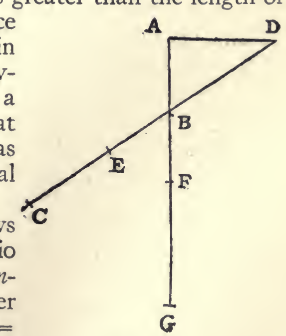

Fig. 72